REAL ALGEBRAIC GEOMETRY LECTURE NOTES (17: 15/12/09)

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Contents

Let R be a real closed field.

1. Decomposition of semialgebraic sets

In the last lecture we proved the following:

Proposition 1.1. *(Main proposition "with coefficients")*
Let $f_1(\underline{\mathrm{x}}, \mathrm{y}), \dots, f_s(\underline{\mathrm{x}}, \mathrm{y})$ *be polynomials in* $n+1$ *variables with coefficients in* R. *Let* $q := \max_{i=1,\dots,s}\{\deg \text{ in y of } f_i(\underline{\mathrm{x}}, \mathrm{y})\}$ *and* $w \in W_{s,q}$.

Then there exists a boolean combination $B_w(\underline{\mathrm{x}})$ *of polynomial equations and inequalities in the variables* $\underline{\mathrm{x}}$ *with coefficients in* R *such that for any* $\underline{x} \in R^n$,

$$\mathrm{sign}_R(f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})) = w \;\Leftrightarrow\; B_w(\underline{x}) \text{ is satisfied in } R.$$

Theorem 1.2. *Let* $\underline{\mathrm{x}} = (\mathrm{x}_1, \dots, \mathrm{x}_n)$. *Let* $f_1(\underline{\mathrm{x}}, \mathrm{y}), \dots, f_s(\underline{\mathrm{x}}, \mathrm{y})$ *be polynomials in* $n+1$ *variables with coefficients in* R. *Then there exixts a partition of* R^n *into a finite number of semialgebraic sets*

$$R^n = A_1 \,\dot\cup\, \cdots \,\dot\cup\, A_m$$

and for each $i = 1, \dots, m$ *there exists a finite number (possibly* 0*) of continuous semialgebraic functions* $\xi_{i1}, \dots, \xi_{il_i}$ *defined on* A_i *with*

$$\xi_{i1} < \cdots < \xi_{il_i}$$

$$\xi_{ij} \colon A_i \longrightarrow R$$

and $\xi_{ij}(\underline{x}) < \xi_{ij+1}(\underline{x})$ *for all* $\underline{x} \in A_i$, *for all* $j = 1, \dots, l_i$, *such that*

(i) *for each* $\underline{x} \in A_i$, $\{\xi_{i1}(\underline{x}), \dots, \xi_{il_i}(\underline{x})\} = \{$*roots of those polynomials among* $f_1(\underline{x}, \mathrm{y}), \dots, f_s(\underline{x}, \mathrm{y})$ *which are not identically zero*$\}$;

(ii) *for each* $\underline{x} \in A_i$ *and* $y \in R$, $\mathrm{sign}(f_1(\underline{x}, y)), \dots, \mathrm{sign}(f_s(\underline{x}, y))$ *depend only on* $\mathrm{sign}(y - \xi_{i1}), \dots, \mathrm{sign}(y - \xi_{il_i})$.

Definition 1.3. Let $f_1(\underline{\mathrm{x}},\mathrm{y}),\ldots,f_s(\underline{\mathrm{x}},\mathrm{y})$ be polynomials in $n+1$ variables with coefficients in R. A partition of R^n into a finite number of semialgebraic sets

$$R^n = A_1 \,\dot{\cup}\, \cdots \,\dot{\cup}\, A_m$$

together with continuous semialgebraic functions

$$\xi_{i1} < \cdots < \xi_{il_i} \colon A_i \;\longrightarrow\; R$$

satisfying properties (i) and (ii) of Theorem 1.2 is called a **slicing** of $f_1,\ldots,f_s$ and is denoted by

$$(A_i \;;\; (\xi_{ij})_{j=1,\ldots,l_i})_{i\in\{1,\ldots,m\}}$$

If the $A_1,\ldots,A_m$ are given by boolean conbinations on the polynomials $g_1,\ldots,g_t \in R[\mathrm{x}_1,\ldots,\mathrm{x}_n]$, we say that the $g_1,\ldots,g_t$ **slice** the $f_1,\ldots,f_s$.

Lemma 1.4. *Let $f_1(\underline{\mathrm{x}},\mathrm{y}),\ldots,f_s(\underline{\mathrm{x}},\mathrm{y})$ be polynomials in $R[\underline{\mathrm{x}},\mathrm{y}]$ and $(A_i \;;\; (\xi_{ij})_{j=1,\ldots,l_i})_{i\in\{1,\ldots,m\}}$ a slicing of $f_1,\ldots,f_s$. Then for every i, $1\leqslant i\leqslant m$, and every j, $0\leqslant j\leqslant l_i$, the slice*

$$]\xi_{ij},\xi_{ij+1}[\; := \{(\underline{x},y)\in R^{n+1} : \underline{x}\in A_i \text{ and } \xi_{ij}(\underline{x}) < y < \xi_{ij+1}(\underline{x})\}$$

is semialgebraic and semialgebraically homeomorphic to $A_i \times\,]0,1[$ (with the convention $\xi_{i0}=-\infty$ and $\xi_{il_{i+1}}=+\infty$).

Proof. Each slice is semialgebraic, since A_i and the functions ξ_{ij}, $j=1,\ldots,l_i$ are semialgebraic. We now give explicity the semialgebraic homeomorphism

$$h\colon]\xi_{ij},\xi_{ij+1}[\;\longrightarrow\; A_i{\times}]0,1[.$$

For $j=1,\ldots,l_i-1$ define:

$$h(\underline{x},y) = (\underline{x},\;(y-\xi_{ij}(\underline{x}))/(\xi_{ij+1}(\underline{x})-\xi_{ij}(\underline{x}))).$$

For $j=0$, $\xi_{i0}=-\infty$, define (if $l_i\neq 0$):

$$h(\underline{x},y) = (\underline{x},\;(1+\xi_{i,1}(\underline{x})-y)^{-1}).$$

For $j=l_i\neq 0$, $\xi_{il_{i+1}}=+\infty$, define:

$$h(\underline{x},y) = (\underline{x},\;(y-\xi_{i,l_i}(\underline{x})+1)^{-1}).$$

If $l_i=0$, $\xi_0=-\infty$ and $\xi_1=+\infty$, define:

$$h(\underline{x},y) = (\underline{x},\;(y+\sqrt{1+y^2})/2\sqrt{1+y^2}).$$

□

Theorem 1.5. *Every semialgebraic subset of R^n is the disjoint union of a finite number of semialgebraic sets, each of them semialgebraically homeomorphic to an open hypercube $]0,1[^d \subset R^d$, for some $d \in \mathbb{N}$ (where $]0,1[^0$ is a point).*

Proof. By induction on n.

For $n = 1$, we already know that every semialgebraic subset of R is the union of a finite number of points and open intervals. Open intervals are semialgebraically homeomorphic to $]0,1[$ and a point is semialgebraically homeomorphic to $]0,1[^0$.

We now assume that the result holds for n. Let S be a semialgebraic subset of R^{n+1}, given by a boolean combination of sign conditions on the polynomials $f_1, \ldots, f_s$, and let $(A_i\ ;\ (\xi_{ij})_{j=1,\ldots,l_i})_{i \in \{1,\ldots,m\}}$ be a slicing of $f_1, \ldots, f_s$.

By induction, all A_i are semialgebraically homeomorphic to open hypercubes. Moreover, S is the union of a finite number of semialgebraic sets that are either the graph of a function ξ_{ij}, or a slice $]\xi_{ij}, \xi_{ij+1}[$ as in Lemma 1.4.

The graph of ξ_{ij} is semialgebraically homeomorphic to A_i, while, by Lemma 1.4, the slice $]\xi_{ij}, \xi_{ij+1}[$ is semialgebraically homeomorphic to $A_i \times]0,1[$. □