# REAL ALGEBRAIC GEOMETRY LECTURE NOTES (12 Continued: 26/11/2009)

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## THE TARSKI-SEIDENBERG PRINCIPLE

**Recall.** Let $R$ be a real closed field, $a \in R$. Define

$$sign(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

The Tarski-Seidenberg Principle is the following result.

**Theorem 1.** Let $f_i(\underline{T}, X) = h_{i,m_i}(\underline{T})X^{m_i} + \ldots + h_{i,0}(\underline{T})$ for $i = 1, \ldots, s$ be a sequence of polynomials in n+1 variables $(\underline{T} = (T_1, \ldots, T_n), X)$ with coefficients in $\mathbb{Z}$. Let $\epsilon$ be a function from $\{1, \ldots, s\}$ to $\{-1, 0, 1\}$. Then there exists a finite boolean combination $B(\underline{T}) := S_1(\underline{T}) \vee \ldots \vee S_p(\underline{T})$ of polynomial equations and inequalities in the variables $T_1, \ldots, T_n$ with coefficients in $\mathbb{Z}$ such that for every real closed field R and for every $\underline{t} \in R^n$, the system

$$\begin{cases} sign(f_1(\underline{t}, X)) = \epsilon(1) \\ \qquad\vdots \\ sign(f_s(\underline{t}, X)) = \epsilon(s) \end{cases}$$

has a solution $x \in R$ if and only if $B(\underline{t})$ holds true in R.

**Notation I.** Let $f_1(X), \ldots, f_s(X)$ be a sequence of polynomials in R[X]. Let $x_1 < \ldots < x_N$ be the roots in R of all $f_i$ that are not identically zero.

Set $x_0 := -\infty$ , $x_{N+1} := +\infty$

**Remark 1.** Let $m := max(deg f_i, i = 1, \ldots, s)$. Then $N \leq sm$.

Set $I_k := ]x_k, x_{k+1}[$ , $k = 0, \ldots, N$

**Remark 2.** $sign(f_i(x))$ is constant on $I_k$, for each $i \in 1, \ldots, s$, for each $k \in 0, \ldots, N$.

Set $sign(f_i(I_k)) := sign(f_i(x)),\ x \in I_k$

**Notation II.** Let $SIGN_R(f_1, \dots, f_s)$ be the matrix with $s$ rows and $2N+1$ columns whose $i^{th}$ row (for $i = \{1, \dots, s\}$) is

$$sign\big(f_i(I_0)\big), sign\big(f_i(x_1)\big), sign\big(f_i(I_1)\big), \dots, sign\big(f_i(x_N)\big), sign\big(f_i(I_N)\big).$$

i.e. $SIGN_R(f_1, \dots, f_s)$ is an $s\times(2N+1)$ matrix with coefficients in $\{-1, 0, 1\}$ and

$$SIGN_R(f_1, ..., f_s) := \begin{pmatrix} sign f_1(I_0) & sign f_1(x_1) & \dots & sign f_1(x_N) & sign f_1(I_N) \\ sign f_2(I_0) & sign f_2(x_1) & \dots & sign f_2(x_N) & sign f_2(I_N) \\ \vdots & \vdots & & \vdots & \vdots \\ sign f_s(I_0) & sign f_s(x_1) & \dots & sign f_s(x_N) & sign f_s(I_N) \end{pmatrix}$$

**Remark 3.** Let $f_1, \dots, f_s \in R[X]$ and $\epsilon : \{1, \dots, s\} \to \{-1, 0, +1\}$. The system

$$\begin{cases} sign(f_1(X)) = \epsilon(1) \\ \vdots \\ sign(f_s(X)) = \epsilon(s) \end{cases}$$

has a solution $x \in R$ if and only if one column of $SIGN_R(f_1, \dots, f_s)$ is precisely the matrix $\begin{bmatrix} \epsilon(1) \\ \vdots \\ \epsilon(s) \end{bmatrix}$.

**Notation III.** Let $M_{P\times Q}$ := the set of $P\times Q$ matrices with coefficients in $\{-1, 0, +1\}$.

Set $W_{s,m}$ := the disjoint union of $M_{s\times(2l+1)}$, for $l = 0, \dots, sm$.

**Notation IV.** Let $\epsilon : \{1, \dots, s\} \to \{-1, 0, 1\}$. Set

$$W(\epsilon) = \{M \in W_{s,m} : one\ column\ of\ M\ is\ \begin{bmatrix} \epsilon(1) \\ \vdots \\ \epsilon(s) \end{bmatrix}\} \subseteq W_{s,m}$$

**Lemma 2.** (Reformulation of remark 3 using notation IV) Let $\epsilon : \{1,\dots,s\} \to \{-1,0,+1\}$, $R$ real closed field and $f_1(X),\dots,f_s(X) \in R[X]$ of degree $\leq m$. Then the system

$$\begin{cases} sign(f_1(X)) = \epsilon(1) \\ \vdots \\ sign(f_s(X)) = \epsilon(s) \end{cases}$$

has a solution $x \in R$ if and only if $SIGN_R(f_1,\dots,f_s) \in W(\epsilon)$.

By Lemma 2 (setting $W' = W(\epsilon)$), we see that the proof of Theorem 1 reduces to showing the following proposition:

**Main Proposition 3.** Let $f_i(\underline{T},X) := h_{i,m_i}(\underline{T})X^{m_i} + \dots + h_{i,0}(\underline{T})$ for $i = 1,\dots,s$ be a sequence of polynomials in n+1 variables with coefficients in $\mathbb{Z}$, and let $m := max\{m_i | i = 1,\dots,s\}$. Let $W'$ be a subset of $W_{s,m}$. Then there exists a boolean combination $B(\underline{T}) = S_1(\underline{T}) \vee \dots \vee S_p(\underline{T})$ of polynomial equations and inequalities in the variables $\underline{\mathrm{T}}$ with coefficients in $\mathbb{Z}$, such that, for every real closed field R and every $\underline{t} \in R^n$, we have

$$SIGN_R(f_1(\underline{t},X),\dots,f_s(\underline{t},X) \in W' \Leftrightarrow B(\underline{t}) \text{ holds true in R.}$$

The proof of the main Proposition will follow by induction from the next main lemma, where we will show that $SIGN_R(f_1,\dots,f_s)$ is completely determined by the "$SIGN_R$"of a (possibly) longer but simpler sequence of polynomials, i.e. $SIGN_R(f_1,\dots,f_{s-1},f_s',g_1,\dots,g_s)$, where $f_s'$ = the derivative of $f_s$, and $g_1,\dots,g_s$ are the remainders of the euclidean division of $f_s$ by $f_1,\dots,f_{s-1},f_s'$, respectively.

First we will state and prove the lemma and then prove the proposition.