REAL ALGEBRAIC GEOMETRY LECTURE NOTES (03: 27/10/09)

SALMA KUHLMANN

Contents

1. Preorderings and positive cones

Definition 1.1. (*Präordnung*) Let K be a field and let $T \subseteq K$ such that

(i) $T + T \subseteq T$,
(ii) $TT \subseteq T$,
(iii) $a^2 \in T$ for every $a \in K$.

(where $T + T := \{t_1 + t_2 : t_1, t_2 \in T\}$ and $TT := \{t_1 t_2 : t_1, t_2 \in T\}$).
Then T is said to be a **preordering** (or **cone**) of K.

Definition 1.2. (*echte Präordnung*) A preordering T of a field K is said to be **proper** if $-1 \notin T$.

Definition 1.3. (*Positivkegel*) A proper preordering T of a field K is said to be a **positive cone** if $-T \cup T = K$, where $-T := \{-t : t \in T\}$.

Proposition 1.4. *Let $(K, \leqslant)$ be an ordered field. Then the set*

$$P := \{x \in K : x \geqslant 0\}$$

is a positive cone of K. Conversely, if P is a positive cone of a field K, then $\forall\, x, y \in K$

$$x \leqslant y \;\Leftrightarrow\; y - x \in P$$

defines an ordering on K such that $(K, \leqslant)$ is an ordered field.

Therefore for every field K there is a bijection between the set of the orderings on K and the set of the positive cones of K.

Notation 1.5. Let K be a field. We denote by $\sum K^2$ the set

$$\{a_1^2 + \cdots + a_n^2 : n \in \mathbb{N},\, a_i \in K,\, i = 1, \ldots, n\}.$$

Exercise 1.6. Let K be a field. Then

(1) $\sum K^2$ is a preordering of K.

(2) $\sum K^2$ is the smallest preordering of K, i.e. if T is a preordering of K, then $\sum K^2 \subseteq T$.

(3) If K is real then $-1 \notin \sum K^2$ (i.e. $\sum K^2$ is a proper preordering).

(4) If K is algebraically closed then it is not real.

(5) Let (K, P) be an ordered real field, F a field and

$$\varphi : F \longrightarrow K$$

an homomorphism of fields. Then $Q := \varphi^{-1}(P)$ is an ordering of F (Q is said to be the **pullback** of P).

(6) If P, Q are positive cones of K with $P \subseteq Q$, then $P = Q$.

(7) In particular, if $\sum K^2$ is a positive cone (or ordering: see 1.4) of K, then it is the unique ordering of K.

Remark 1.7. Let K be a field with $\mathrm{char}(K) \neq 2$. If $T \subseteq K$ is a preordering which is not proper (i.e. $-1 \in T$), then $T = K$.

Proof. For every $x \in K$,

$$x = \left(\frac{x+1}{2}\right)^2 + (-1)\left(\frac{x-1}{2}\right)^2 \in T.$$

□

Remark 1.8. Let $\mathcal{T} = \{T_i : i \in I\}$ be a family of preorderings of a field K. Then

(i)

$$\bigcap_{i \in I} T_i$$

is a preordering of K.

(ii) if $\forall\, i, j \in I\ \exists k \in I$ such that $T_i \cup T_j \subseteq T_k$, then

$$\bigcup_{i \in I} T_i$$

is a preordering of K.

2. A crucial Lemma

Lemma 2.1. *Let K be a field and T a proper preordering of K. If $a \in K$ and $a \notin T$, then*

$$T - aT = \{t_1 - at_2 : t_1, t_2 \in T\}$$

is a proper preordering of K.

Proof. Since $K^2 \subseteq T$, also $K^2 \subseteq T - aT$. Clearly $(T - aT) + (T - aT) \subseteq T - aT$. Moreover $\forall\, t_1, t_2, t_3, t_4 \in T$,

$$(t_1 - at_2)(t_3 - at_4) = t_1t_3 + a^2t_2t_4 - a(t_1t_4 + t_2t_3) \ \in \ T - aT,$$

therefore $(T - aT)(T - aT) \subseteq (T - aT)$ and $T - aT$ is a preordering of K.

If $(T - aT)$ is not proper, then $-1 = t_1 - at_2$ for some $t_1, t_2 \in T$ with $t_2 \neq 0$, since T is proper. Therefore

$$a = \frac{1}{t_2^2}(1 + t_1)t_2 \ \in \ T,$$

contradiction. □

3. Several consequences

Corollary 3.1. *Every maximal proper preordering of a field K is an ordering (positive cone: see 1.4) of K.*

Corollary 3.2. *Every proper preordering of a field K is contained in an ordering of K.*

Proof. Let T be a proper preordering. Let

$$\mathcal{T} = \{\, T' : T' \supseteq T,\ T' \text{ is a proper preordering of } K \,\}.$$

$\mathcal{T}$ is non-empty and for every ascending chain of $\mathcal{T}$

$$T_{i_1} \ \subseteq \ T_{i_2} \ \subseteq \ \ldots \ \subseteq \ T_{i_k} \ \subseteq \ \ldots$$

by 1.8(ii) $\bigcup T_{i_j}$ is a proper preordering containing T and Zorn's Lemma applies.

Let P be a maximal element of $\mathcal{T}$. Then P is a maximal preordering of K containing T, and by 3.1 P is an ordering.

□

Corollary 3.3. *Let T be a proper preordering of a field K. Then*

$$T = \bigcap \{P : T \subseteq P,\ P \textit{ positive cone of } K\}.$$

($\subseteq$) It is obvious.

($\supseteq$) Let $a \in K$ such that a is contained in every positive cone containing T. If $a \notin T$, then by Lemma 2.1 $T - aT$ is a proper preordering of K. By Corollary 3.2, $T - aT$ is contained in a positive cone P of K. Then $-a \in P$ and $a \notin P$.

Corollary 3.4. *(Characterization of real fields) Let K be a field. The following are equivalent:*

(1) *K is real (i.e. K has an ordering).*

(2) *K has a proper preordering.*

(3) *$\sum K^2$ is a proper preordering (i.e. $-1 \notin \sum K^2$).*

(4) *$\forall\, n \in \mathbb{N}$, $\forall\, a_1, \dots, a_n \in K$*

$$\sum_{i=1}^{n} a_i^2 = 0 \;\Rightarrow\; a_1 = \cdots = a_n = 0.$$

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) obvious. We show now (3) $\Leftrightarrow$ (4).

($\Rightarrow$) Let $\sum_{i=1}^{n} a_i^2 = 0$ and suppose $a_i \neq 0$ for some $1 \leqslant i \leqslant n$. Say $a_n \neq 0$. Then

$$\frac{a_1^2 + \cdots + a_n^2}{a_n^2} \;=\; 0,$$

and

$$\left(\frac{a_1}{a_n}\right)^2 + \cdots + \left(\frac{a_{n-1}}{a_n}\right)^2 + 1 = 0.$$

Therefore $-1 \in \sum K^2$, contradiction.

($\Leftarrow$) Suppose $-1 \in \sum K^2$, so

$$-1 \;=\; b_1^2 + \cdots + b_s^2,$$

for some $s \in \mathbb{N}$ and $b_1, \dots, b_s \in K$. Then

$$1 + b_1^2 + \cdots + b_s^2 = 0$$

and $1 = 0$, contradiction.

To complete the proof note that if $-1 \notin \sum K^2$ then $\sum K^2$ is a proper preordering, and by Corollary 3.2 K has an ordering. This proves (3) $\Rightarrow$ (1). □

Corollary 3.5. *(Artin) Let K be a real field. Then*

$$\sum K^2 = \bigcap \{P : P \text{ is an ordering of } K\}.$$

In other words, if K is a real field and $a \in K$, then

$$a \geqslant_P 0 \;\text{ for every ordering } P \;\Leftrightarrow\; a \in \sum K^2.$$