Hardy fields and generalized power series with exponential.

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Real exponential fields.

$\mathbb{K} = (\mathbb{K}, +, \cdot, 0, 1, <, \exp)$ real field endowed with an **exponential map**

$$\exp : (\mathbb{K}, +, <) \twoheadrightarrow (\mathbb{K}_{>0}, \cdot, <)$$

Possibly with a **derivation**:

$$\partial : (\mathbb{K}, +, <) \to (\mathbb{K}, +, <)$$
$$\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b)$$
$$\partial(\exp(a)) = \partial(a) \cdot \exp(a)$$

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Real exponential fields.

Key objects:

Archimedean: $(\mathbb{R}, \exp)$, its prime model

Non archimedean: **exponential Hardy fields**, **exponential (sub)fields of generalized series**

Universal domain: **surreal numbers**

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Hardy fields.

$\mathcal{G}$ = ring of germs at $+\infty$ of real functions

Definition (Bourbaki 76)

A **Hardy field** $\mathbb{K}$ is a subring of $\mathcal{G}$ which is a *field* and which is *closed under derivation*.

$\Rightarrow$ strongly non-oscillating: $\mathbb{K}$ ordered subfield of $\bigcap_{k\in\mathbb{N}_0}\mathcal{C}^k$

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Example

- $\mathbb{R} \subseteq \mathbb{R}(t) \subseteq \mathbb{R}(\log(t), t, \exp(t)) \subseteq \cdots$

- Log-exp functions = "L-functions" (Hardy 12)

- Unary definable functions in an **o-minimal structure** expanding $\mathbb{R}$: e.g. $\mathbb{R}_{\exp}$ (Wilkie 96), $\mathbb{R}_{\mathrm{an},\exp}$ (van den Dries-Miller 94),...

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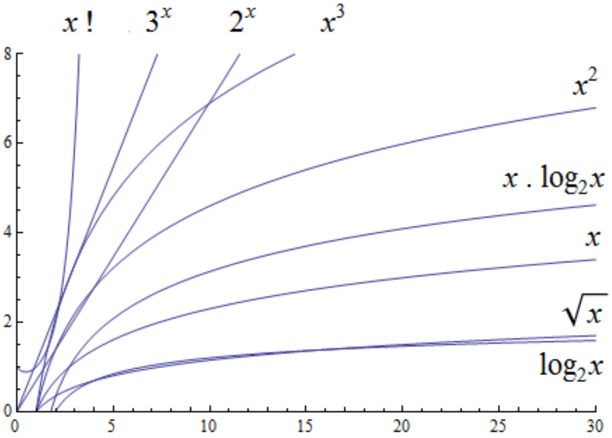
Asymptotic scales



The natural valuation.

Definition

Valuation v with valuation ring $\mathcal{O}_v = \mathrm{Conv}(\mathbb{Q})$

- value group $\Gamma := \{archimedean \ \sim \text{classes of } \mathbb{K}\}$
- residue field $\subseteq \mathbb{R}$
- **rank** $\Phi := \{\text{archimedean } \sim \text{classes of } \Gamma\}$

Dominance relation: $a \preccurlyeq b :\Leftrightarrow v(a) \geq v(b)$

Extensions of Hardy fields

Still get a Hardy field by:

- Passing to the **real closure** (Bourbaki, A. Robinson 72)

- Closing under **real powers** (Rosenlicht 83)

- Adjoining solutions of *certain* differential equations (Boshernitzan, Rosenlicht, Singer 80's)

  → (linear) order 1: **Liouville closure: integration, exp and log**; other e.g. $P(y)y' = Q(y)$

  → certain linear order 2 diff equ

  and much more: **DIVP** (Aschenbrenner-van den Dries-van der Hoeven preprint 2023)

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Extensions of Hardy fields

• O-minimal expansions of $\mathbb{R}$: adjoining **quasi-analytic classes** (Rolin-Speissegger-Wilkie 2003); passing to the **pfaffian closure** (Speissegger 2018)

• Adjoining **quasianalytic IlYashenko algebras** (link with *Hilbert 16*: Speissegger, Galal-Kaiser-Speissegger 2020)

• Adjoining *certain* solutions of functional equations: **transexponential function** (Boshernitzan 86)

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Hardy fields as differential valued fields

Hardy type derivation (Rosenlicht 79-83, Kuhlmann-M. 2011):

- (HD1) $\mathbb{R} = \ker \partial$
- (HD2) **l'Hospital's rule**:

$$\forall v(a), v(b) \neq 0, v(a) \geq v(b) \Leftrightarrow v(\partial(a)) \preccurlyeq v(\partial(b))$$

- (HD3) **logarithmic derivative**:

$$|v(a)| \gg |v(b)| > 0 \Leftrightarrow v(\partial(a)/a) < v(\partial(b)/b)$$

H-**field** (Aschenbrenner-van den Dries 2002):

- (HF1) = (HD1)
- $a > \mathbb{R} \Rightarrow a' > 0$

Exponential Hardy fields

Valuative properties:

• Natural valuation ⇒ **Poincaré asymptotic expansions** (Rosenlicht 83)

• **Exponential rank** (Kuhlmann-Kuhlmann 2000)

• **Levels** (Rosenlicht 87, Marker-Miller 97, Kuhlmann-Kuhlmann 2003)

Exponential Hardy fields

Differential and "analytic" properties:

- Compatibility

$$\partial(\exp(a)) = \partial(a) \cdot \exp(a) \qquad \partial(\log(a)) = \frac{\partial(a)}{a}$$

- "Strong" morphism

$$\exp(\sum \alpha u) = \prod \exp(u)^{\alpha} \quad \log(\prod u^{\alpha}) = \sum \alpha \log(u)$$

$\rightsquigarrow$ Composition...

Generalized series

$\mathbb{R}((t^\Gamma))$ where :

- $\Gamma = (\Gamma, +, <)$ ordered abelian group of **exponents**
- $\mathbb{R}$ as ordered field of **coefficients**.

Definition (Hahn 07)

A **generalized series** is

$$a : \Gamma \to \mathbb{R}$$

with well-ordered support:

$$\mathrm{supp}(a) := \{\gamma \in \Gamma \ : \ a(\gamma) \neq 0\}$$

Notation: $a = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$

The natural valuation.

Definition (Baer 27)

Ordering: $a > 0 \Leftrightarrow a_{\gamma_0} > 0$ where $\gamma_0 := \min(\text{supp } a)$.

$\Rightarrow$ **natural valuation** on $\mathbb{R}((\Gamma))$ is:

$$\begin{array}{rccc} v: & \mathbb{R}((\Gamma)) & \to & \Gamma \cup \{\infty\} \\ & a & \mapsto & \min(\text{supp } a) \\ & 0 & \mapsto & \infty \end{array}$$

The order type of Φ is called the **rank** of Γ, and as well the **rank** of $\mathbb{R}((\Gamma))$.

•Kaplansky 42: *Universal domain for valued fields*

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Hardy type derivations

Lifting principle (Kuhlmann-M. 2011-12):

$$\text{Field: } \textbf{series} \qquad a = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \mapsto \qquad d(a) = \sum_{\gamma \in \Gamma} a_\gamma d(t^\gamma)$$

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$\cup$	$\cup$	
Val group: **monomials**	$t^\gamma = \prod_{\phi \in \Phi} t_\phi^{r_\phi} \mapsto$	$d(t^\gamma) = t^\gamma \sum_{\phi \in \Phi} r_\phi \frac{d(t_\phi)}{t_\phi}$

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Rank: **fundam. mon.**	$t_\phi \mapsto$	$d(t_\phi) \in \mathbb{R}((t^\Gamma))$

Generalized series and exponentiation

We would like:

$$\mathbb{R}((t^{\Gamma^{<0}}))\oplus\mathbb{R}\oplus\mathbb{R}((t^{\Gamma^{>0}})) \underset{\log}{\overset{\exp}{\rightleftarrows}} t^{\Gamma}\otimes\mathbb{R}_{>0}\otimes(1+\mathbb{R}((t^{\Gamma^{>0}})))$$

Generalized series and exponentiation

For:

$$\mathbb{R} \oplus \mathbb{R}((t^{\Gamma_{>0}})) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathbb{R}_{>0} \otimes (1 + \mathbb{R}((t^{\Gamma_{>0}})))$$

use the **analytic** formulas (Alling 87):

$$\exp(r+\varepsilon) = e^r \cdot \sum_n \frac{\varepsilon^n}{n!} \qquad \log(r\cdot(1+\varepsilon)) = \ln(r) + \sum_{n>0} \frac{\varepsilon^n}{n}$$

Generalized series and exponentiation

For

$$? \quad \mathbb{R}((t^{\Gamma_{<0}})) \underset{\log}{\overset{\exp}{\rightleftarrows}} t^{\Gamma} \quad ?$$

Kuhlmann-Kuhlmann-Shelah 97: $\mathbb{R}((t^{\Gamma_{<0}})) \not\simeq t^{\Gamma}$,
hence $\mathbb{R}((t^{\Gamma}))$ *cannot be an exponential field.*

Generalized series and exponentiation

Kuhlmann-Kuhlmann-Shelah 97: $\mathbb{R}((t^{\Gamma}))$ *cannot be an exponential field.*

BUT a *subfield* of an *extended* $\mathbb{R}((t^{\tilde{\Gamma}}))$ may be! (Dahn-Göring 86):

- **Exp-log series** (Kuhlmann-Kuhlmann 97, Kuhlmann-M. 2011), κ-**bounded series** (Kuhlmann-Shelah)
- **Log-exp series** (van den Dries-Macintyre-Marker 97)
- **Transseries** (Ecalle 92, van der Hoeven 97)

Exponential and logarithmic closure of a series field

A two-complementary-steps procedure:

- To get a **pre-logarithmic** series field:

- To get a **pre-exponential** series field:

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A two-complementary-steps procedure:

- To get a **pre-logarithmic** series field: e.g. by **integration**:

  → *asymptotic integration*:

$$\forall a,\ \exists b,\ \partial(b) \sim a$$

  → *ultrametric fixed point theorem*: holds in spherically complete fields (Priess-Crampe-Ribenboim 93), in unions of spher. compl. fields (vdD-M.M., vdH., K., K.-M., B.M., B.K.M.M., ...)

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A two-complementary-steps procedure:

- To get a **pre-logarithmic** series field:
- To get a **pre-exponential** series field: e.g. by **exponential closure**

$\rightarrow$ *exponential extension*:

$$\begin{array}{ccc}
t^{\Gamma^\sharp} & \xrightarrow{\ I^\sharp\ } & \mathbb{R}((t^{\Gamma^\sharp_{<0}})) \\
\cup & \nwarrow \simeq & \cup \\
t^{\Gamma} & \xrightarrow{\ I\ } & \mathbb{R}((t^{\Gamma_{<0}}))
\end{array}$$

$\rightarrow$ *inductive limite procedure*

Differential valued exponential fields

Exponential Hardy fields	Differential exponential subfields of series

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Exponential Hardy fields	$\overset{?}{\hookrightarrow}$	Differential exponential subfields of series

e.g. $\mathcal{H}(\mathbb{R}_{\mathrm{an},\exp}) \hookrightarrow \mathbb{T}$

Differential valued exponential fields

$$\begin{array}{ccc} \text{Exponential} & & \text{Differential exponential} \\ \text{Hardy fields} & \overset{?}{\hookrightarrow} & \text{subfields of series} \\ & \searrow \quad \swarrow & \\ & \text{Surreal numbers} & \end{array}$$

Surreal numbers (Conway 76, Gonshor 86)

Dedekind-von Neumann numbers

$a =< a^L | a^R >$

Full binary tree

$a = + + - - - + - \cdots$

$\nwarrow$ $\nearrow$

Surreal numbers NO

$\downarrow$

ON-bounded series

$$a = \sum_{\alpha<\lambda} r_\alpha t^{a_\alpha} \in \mathbb{R}((t^{\mathrm{NO}}))_{\mathrm{ON}}$$

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As (analytic) exponential fields

Language $\mathcal{L} = \{+, \cdot, 0, 1, <, \exp\}$

Theorem (Gonshor 86, van den Dries-Ehrlich 2001, Ehrlich-Kaplan 2021)

With Gonshor's exponential:

$$(\mathrm{NO}, \exp) \succcurlyeq (\mathbb{R}, \exp)$$

Moreover, NO *is a canonical monster model for* $\mathrm{Th}(\mathbb{R}_{\exp})$.

NB: *Same statement with* $\mathrm{Th}(\mathbb{R}_{\mathrm{an},\exp})$.

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- Exponential Hardy field $\hookrightarrow$ **NO**
- (Differential) exponential subfield of series $\hookrightarrow$ **NO**

Omega-fields

Definition (Berarducci-Kuhlmann-Mantova-M. 2023)

An ordered field endowed with an omega map:

$$\Omega : (\mathbb{K}, +, <) \twoheadrightarrow (t^{\Gamma}, \cdot, <)$$

where Γ is the value group.

Example

- Surreal numbers **NO** (Conway 76)
- κ-bounded series, transseries $\mathbb{T}$, etc. (BKMM. 2023, Berarducci-Freni 2021)

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Theorem (Aschenbrenner-van den Dries-van der Hoeven 2017-19)

NO $\models \mathrm{Th}(\mathbb{T})$, *the latter being model complete.*

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Questions

- Model theory of differential exponential fields, e.g. $\mathbb{T}$, **NO** etc ? (AvdDvdH, Kaplan)

- Model theory of transexponential Hardy fields? O-minimality? Hyperseries? (Bagayoko-van der Hoeven, Kuhlmann-Krapp, Padgett)

Questions

• Differential Kaplansky embedding theorem for Hardy fields? (Kuhlmann-M.)

• Hardy omega-fields? As e.g. maximal Hardy fields? (BKMM.)

• Composition and compatible derivation on **NO**? (BKMM., Bagayoko-van der Hoeven)

Thanks for your attention...

... and have a nice workshop!