Structure of some subgroups of transseries

(joint work with M. Resman)

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What is the structure of the group G? (e.g. are there any **free subgroups** inside G)

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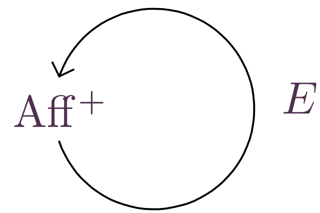

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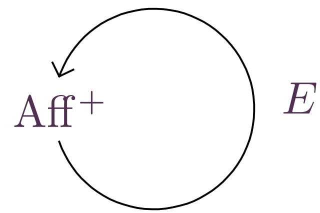

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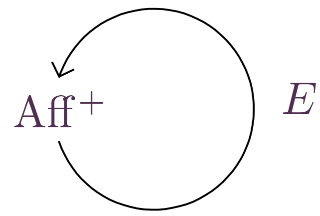

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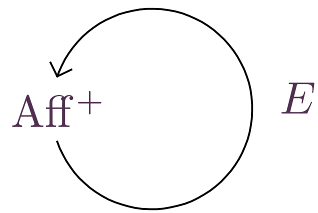

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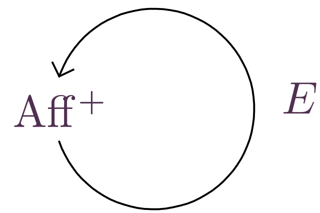

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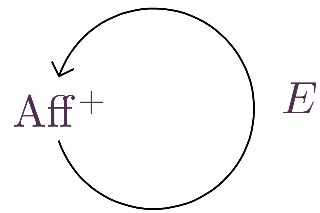

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$$x\to(((x+a_1)^{\lambda_1}+a_2)^{\lambda_2}+\cdots+a_n)^{\lambda_n}$$

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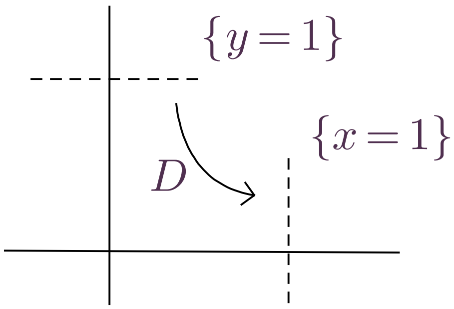

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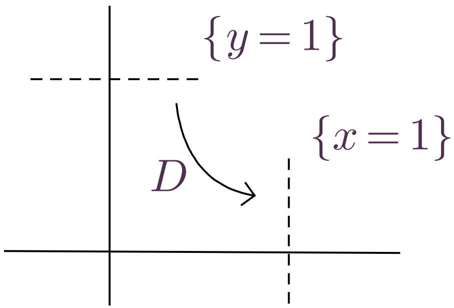

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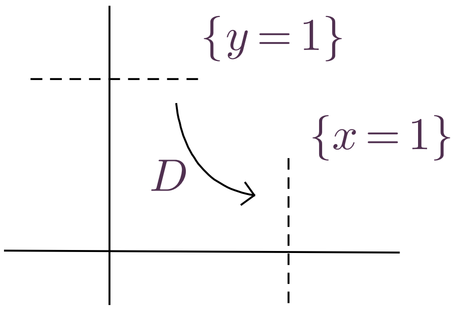

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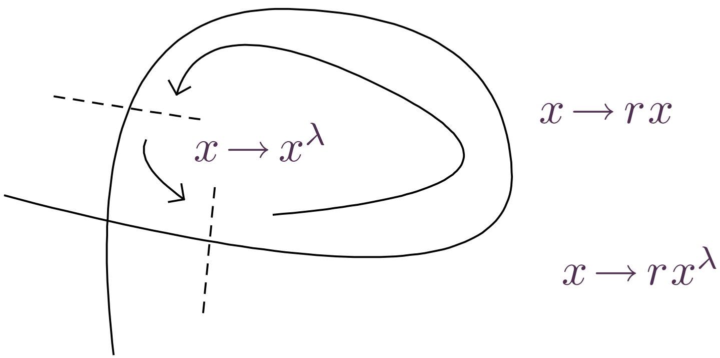

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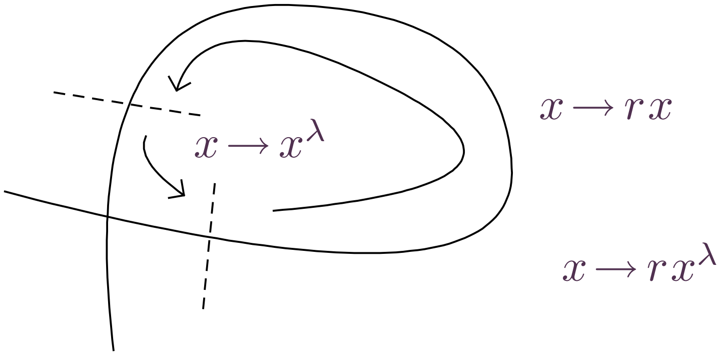

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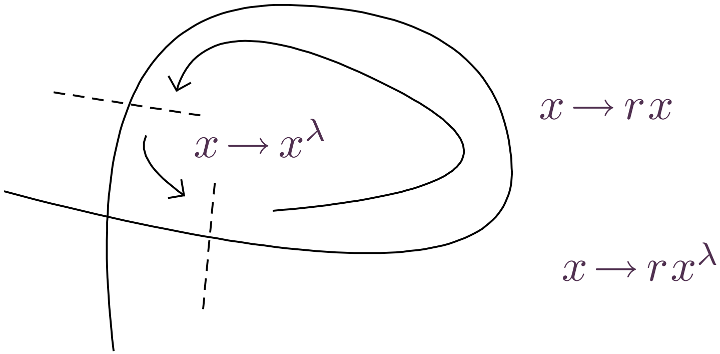

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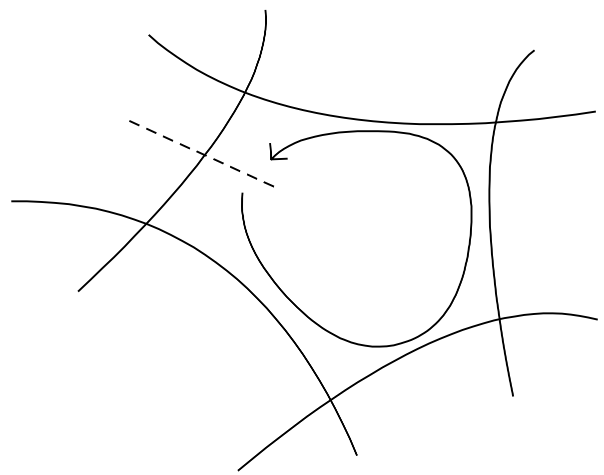

Dulac germ (after Ilyashenko)

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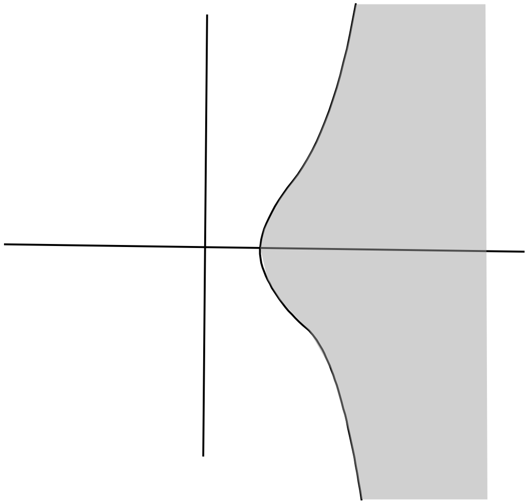

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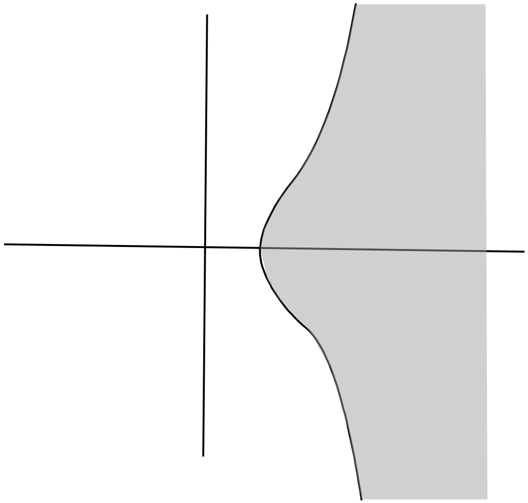

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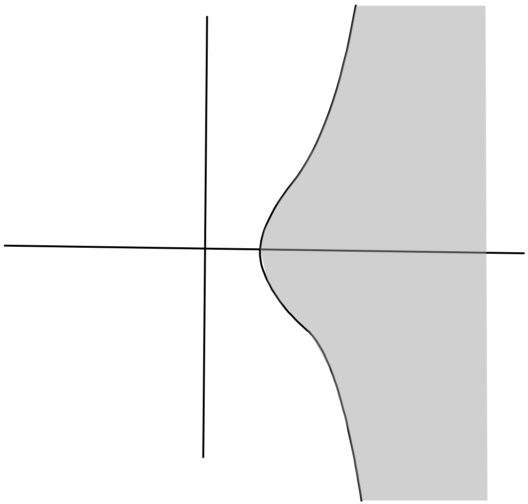

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(Composition) The Dulac germs forms a group under composition, with subgroup

$$\mathcal{D} \cap \mathrm{Homeo}(\mathbb{R}, +\infty)$$

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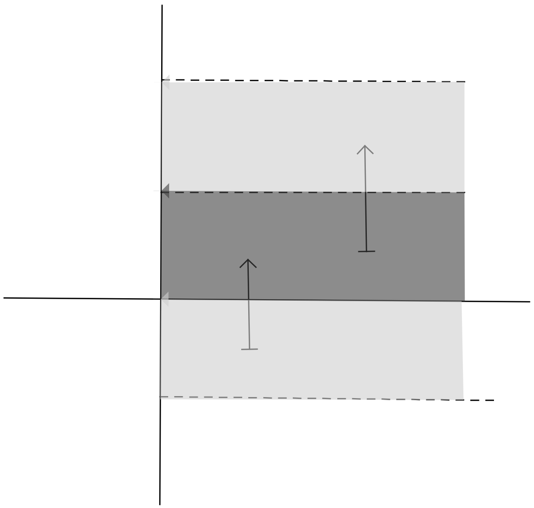

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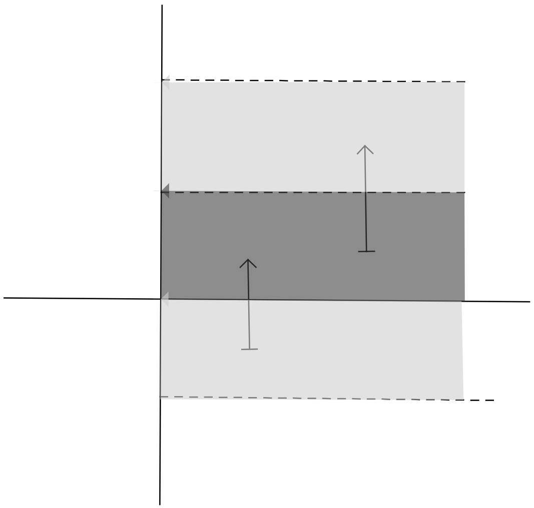

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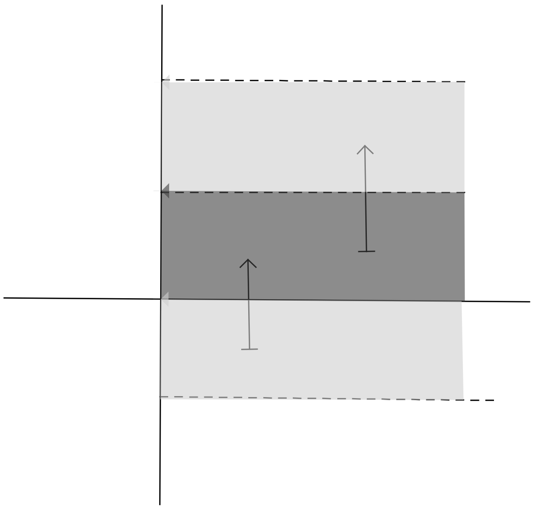

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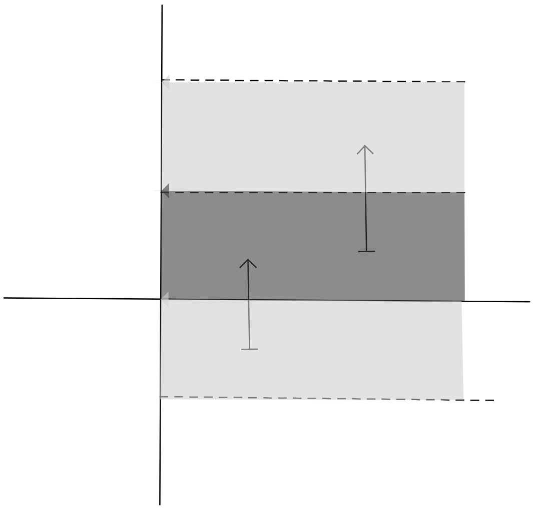

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- d is unramified if and only if there exists a $\varphi \in \mathbb{C}_1\{x\}$ such $d = E^{-1} f E$, where $E(x) = e^{-x}$

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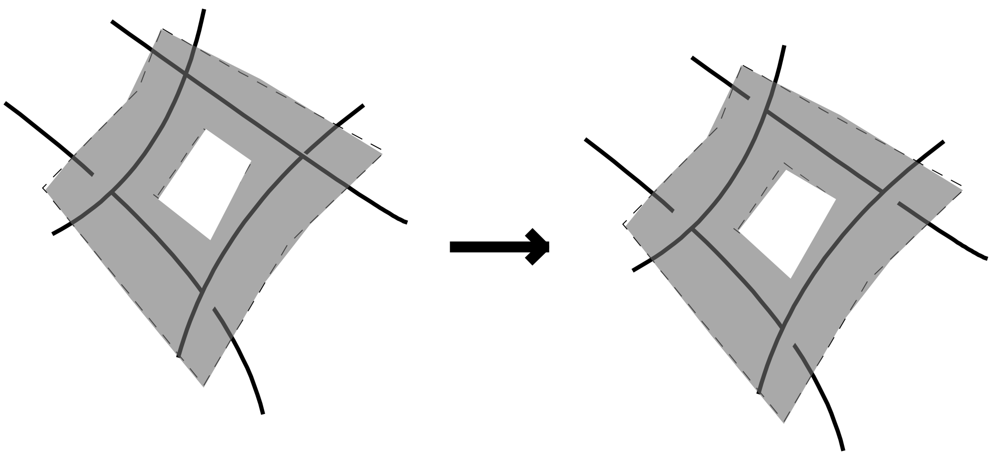

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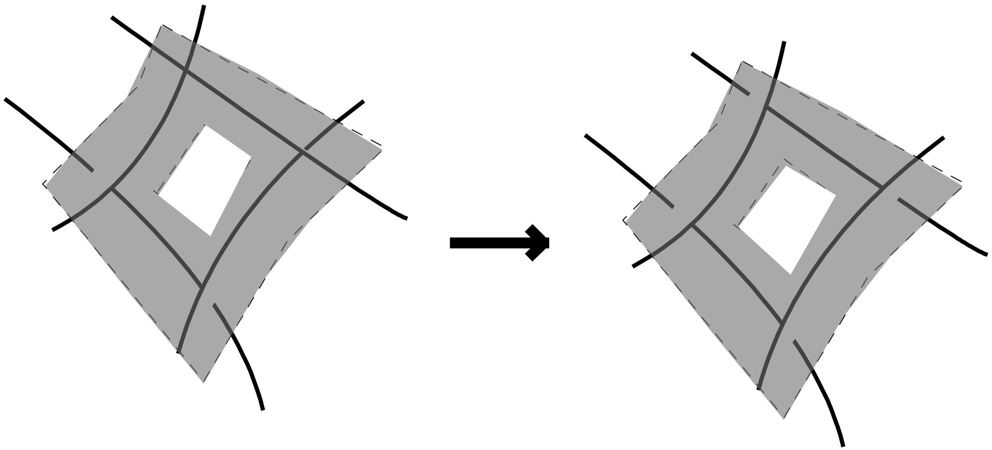

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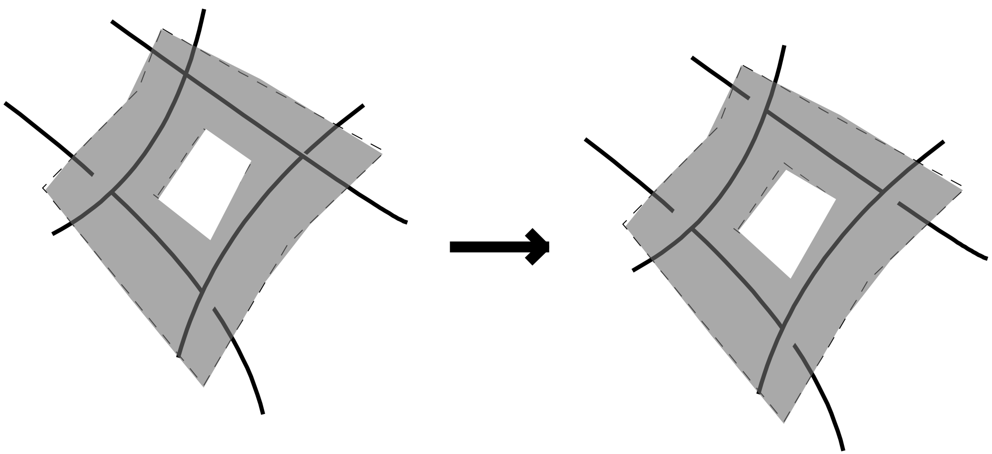

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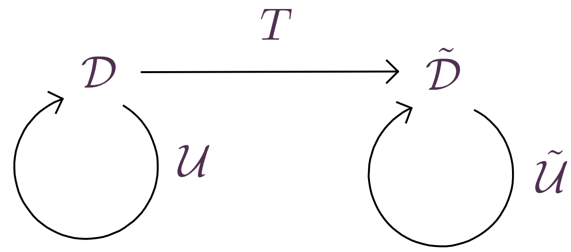
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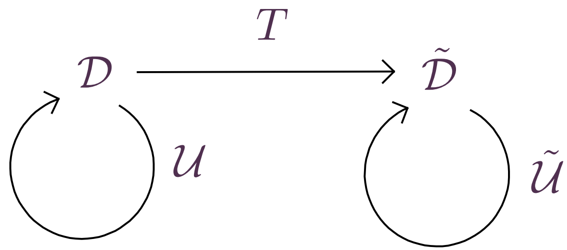

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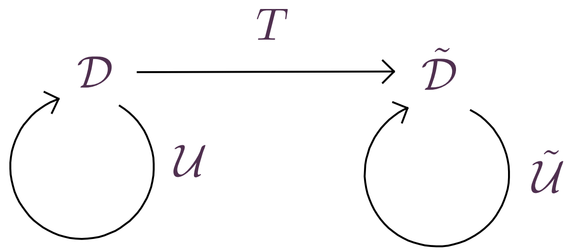

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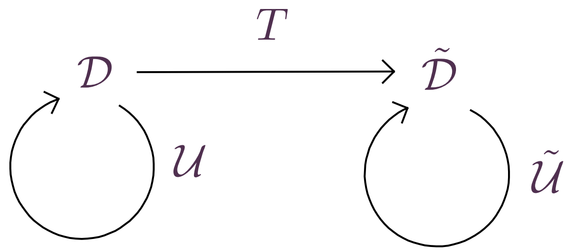

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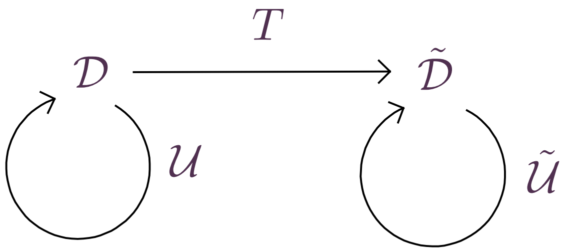

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- **holomorphic classification**: ? (if λ is **not** a Bryuno number there exists germs which are formally conjugated but not analytically conjugated)

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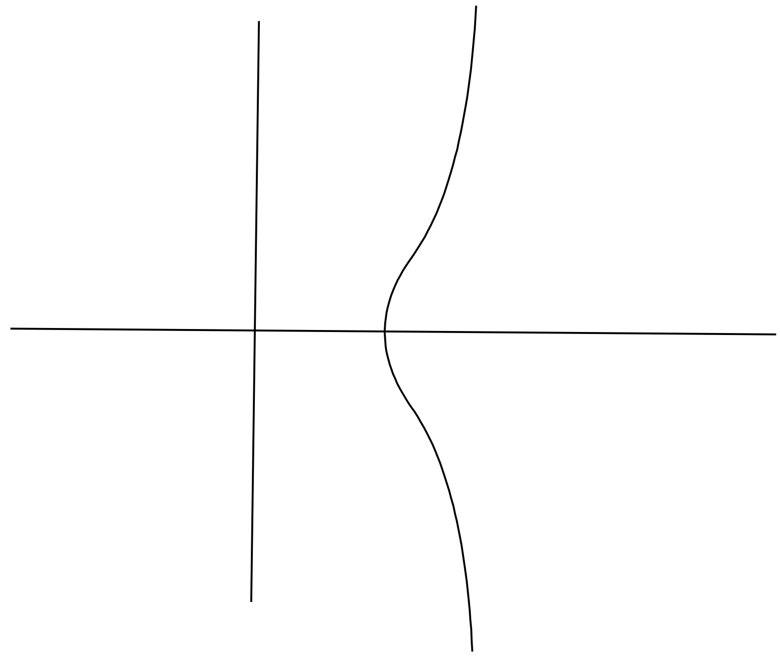
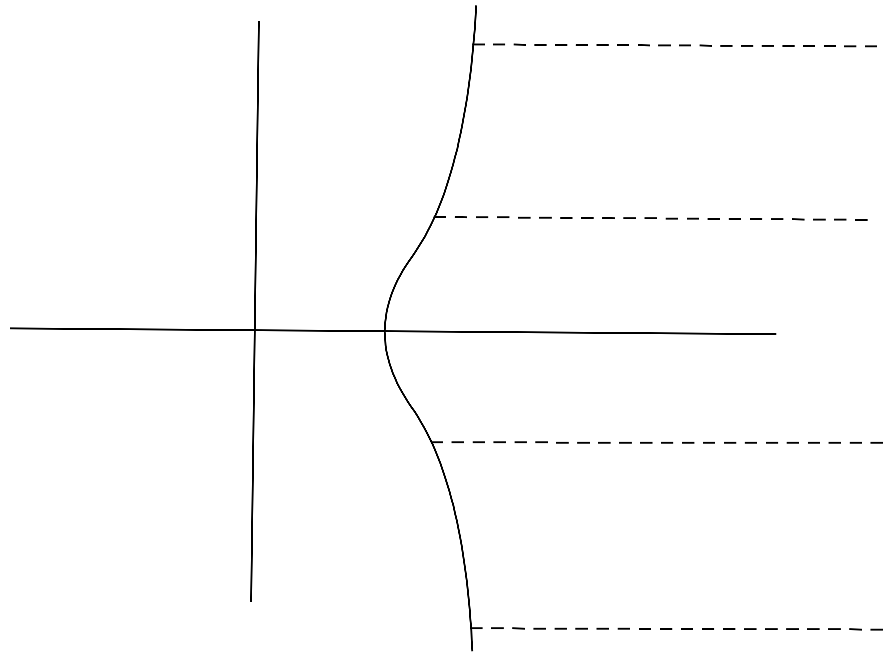
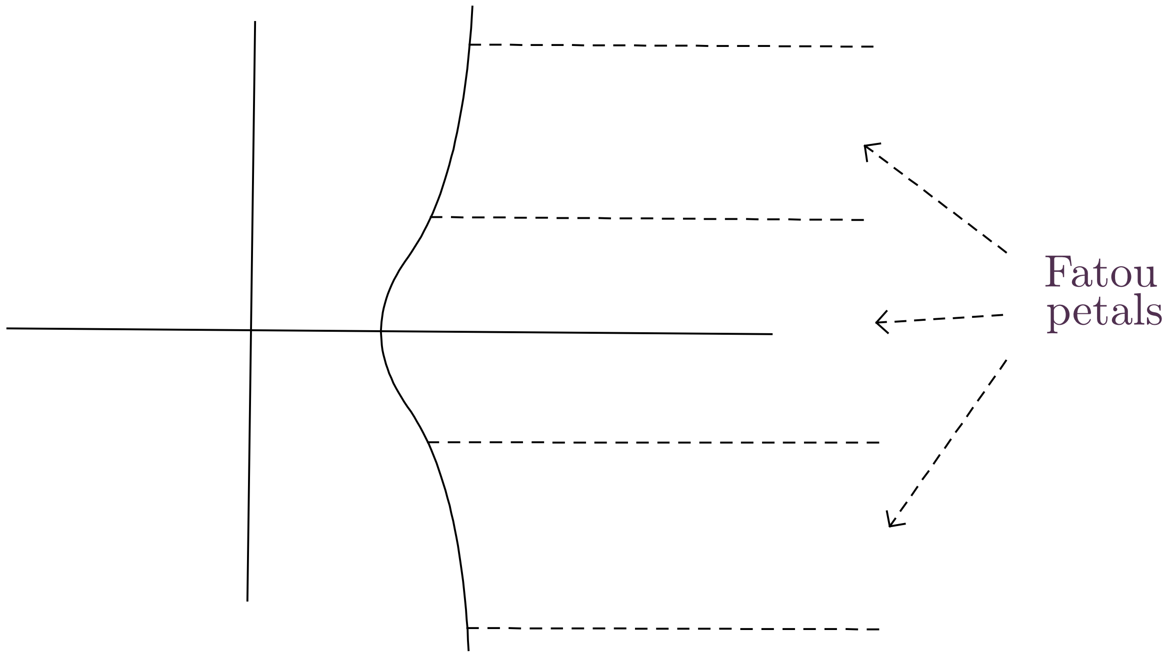
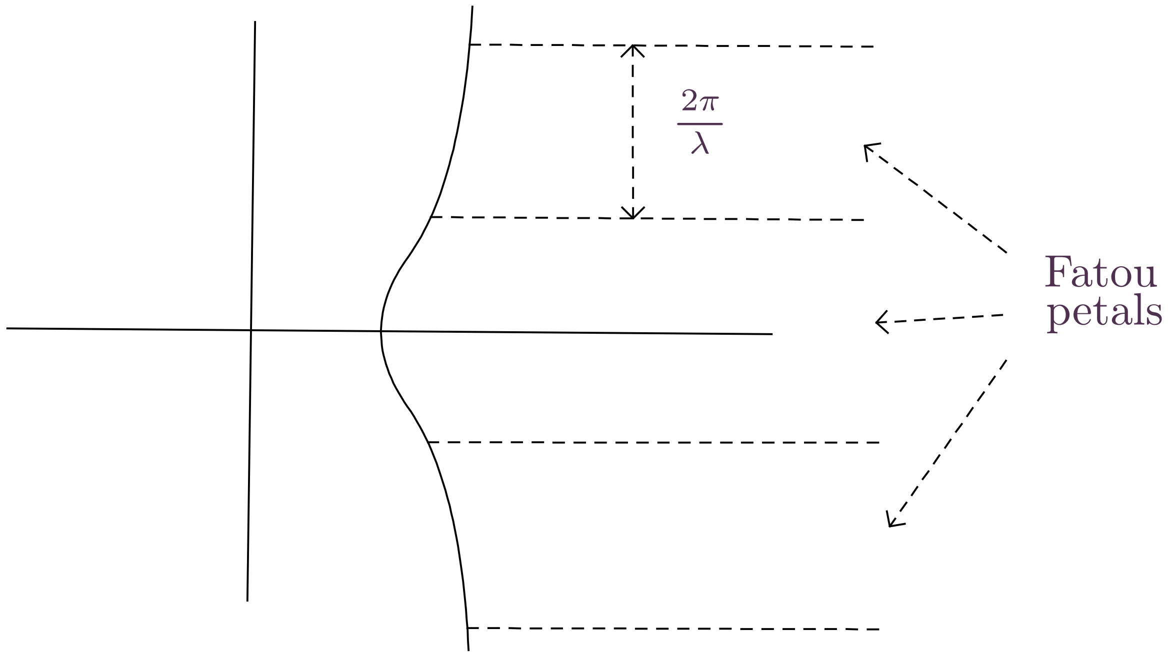
Such conjugacy is possible in strip-like domains...







Fatou
petals


$\frac{2\pi}{\lambda}$
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Thanks for your attention