REAL ALGEBRAIC GEOMETRY LECTURE NOTES (22: 02/07/15)

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1. Decomposition

Recall 1.1. We had the additive and multiplicative decomposition. Let K be a totally ordered field, root closed for positive elements (in particular, if K is real closed). Then

$$(K, +, 0, <) = \mathbb{A} \sqcup \mathbb{A}' \sqcup I_v,$$

$$(K^{>0}, \cdot, 1, <) = \mathbb{B} \sqcup \mathbb{B}' \sqcup 1 + I_v,$$

where $\mathbb{A}$ is a complement to the valuation ring and $\mathbb{A}'$ a complement to the valuation ideal in the valuation ring $S(\mathbb{A}) = [G^{<0}, \{(\overline{K}, +, 0, <)\}]$, $\mathbb{A} \cong (\overline{K}, +, 0, <)$.

$\mathbb{B}$ is a (multiplicative) complement to $U_v^{>0}$ in $K^{>0}$ and $\mathbb{B}'$ is a complement to $1 + I_v$ in U_v. We have $\mathbb{B} \cong G$ and $\mathbb{B}' \cong (\overline{K}^{>0}, \cdot, 1, <)$.

2. Compatible exponentials

Definition 2.1. Let K be a totally ordered field root closed for positive elements.

(*i*) $f : (K, +, 0, <) \xrightarrow{\sim} (K^{>0}, \cdot, 1, <)$ is called an **exponential**.

(*ii*) An exponential f on K is called v**-compatible** (i.e. compatible with the natural valuation) if
- $f(K_v) = U_v^{>0}$ (the image of the valuation ring is the group of positive units)
- $f(I_v) = 1 + I_v$ (the image of the valuation ideal is the group of 1-units)

Remark 2.2. We only study v-compatible exponentials. In fact: a root closed (positive elements) totally ordered field K admits an exponential if and only if it admits a v-compatible exponential.

Indeed, if K admits an exponential e, then it admits a $v-$compatible exponential f, namely: Let $a \in K^{>0}$ such that $e(a) = 2$ and set $f(x) = e(ax)$. One verifies that $f(K_v) = U_v^{>0}$ and $f(I_v) = 1 + I_v$ (ÜA).

The question we want to answer: Given a totally ordered field (root closed for positive elements), when does K admit a v-compatible exponential. We will give necessary conditions on $v(K^*)$ and $\overline{K}$ as follows:

Remark 2.3. If f is a v-compatible exponential, then

(*i*) $f(K_v) = f(\mathbb{A}' \sqcup I_v) = U_v^{>0} = \mathbb{B}' \sqcup 1 + I_v$,

(*ii*) $f(I_v) = 1 + I_v$,

(*iii*) $f(\mathbb{A} \sqcup \mathbb{A}' \sqcup I_v) = \mathbb{B} \sqcup \mathbb{B}' \sqcup (1 + I_v)$.

Therefore f "decomposes" into 3 isomorphisms of ordered groups, namely

- the left exponential $f_L := f \restriction \mathbb{A}$,
- the middle exponential $f_M := f \restriction \mathbb{A}'$,
- the right exponential $f_R := f \restriction I_v$.

Note that

$$\mathbb{A} \sqcup \mathbb{A}' \sqcup I_v \cong (K, +0, <) \cong (K^{>0}, \cdot, 1, <) \cong \mathbb{B} \sqcup \mathbb{B}' \sqcup 1 + I_v$$

and conversely, given $f_L : \mathbb{A} \cong \mathbb{B}$, $f_M : \mathbb{A}' \cong \mathbb{B}'$, and $f_R : I_v \cong 1 + I_v$, the exponential

$$f : (K, +, 0, <) \to (K^{>0}, \cdot, 1, <), a + a' + \varepsilon \mapsto f_L(a) f_M(a') f_R(\varepsilon)$$

on K is v-compatible.

So the question is: when does a totally ordered field K (root closed for positive elements) admit a left expo, a middle expo and a right expo?

Proposition 2.4. *Let K be a non-Archimedean real closed field, $G = v(K^*)$. Assume that K admits a left exponential. Then*

$$S(G) = [G^{<0}, \{(\overline{K}, +, 0, <)\}],$$

i.e. the value set of G is isomorphic to $G^{<0}$ and all Archimedean components of G are isomorphic to $(\overline{K}, +, 0, <)$.

Proof. Note that $\mathbb{A} \cong \mathbb{B}$ and $\mathbb{B} \cong G$, so $\mathbb{A} \cong G$. In particular

$$[G^{<0}, \{(\overline{K}, +, 0, <)\}] = S(\mathbb{A}) = S(G).$$

□

Example 2.5. Consider the divisible ordered abelian group $G = \bigcup_{\mathbb{N}} \mathbb{Q}$ and $\mathbb{K} = \mathbb{R}((G))$. Then $\mathbb{K}$ does not admit an expo because

- G is divisible, so $G^{<0} \not\cong \mathbb{N}$,
- the Archimedean components of G are $\mathbb{Q}$, whereas the residue field is $\mathbb{R}$.

Example 2.6. Consider $G = \bigcup_{\mathbb{Q}} \mathbb{Q}^{\mathrm{rc}}$. Note that the value set of G is $\mathbb{Q}$ and that $G^{<0}$ is a dense linear order without end points. So by Cantor $\mathbb{Q} \cong G^{<0}$.

Consider $\mathbb{K} = \mathbb{Q}^{\mathrm{rc}}((\bigcup_{\mathbb{Q}} \mathbb{Q}^{\mathrm{rc}}))$. Then $\mathbb{K}$ is real closed and also the Archimedean components of G are all isomorphic to $\mathbb{Q}^{\mathrm{rc}}$ (the additive group of the residue field).

Unfortunately $\mathbb{K}$ still does not admit a left exponential because of the following theorem (without proof)

Theorem 2.7. *Let $\mathbb{K} = k((G))$, $G \neq \{0\}$, a real closed field of power series. Then $\mathbb{K}$ does not admit a left exponential function.*

Thus, the necessary condition on the value group is not sufficient.
Question: Does $\mathbb{K} = \mathbb{Q}^{\mathrm{rc}}((\bigcup_{\mathbb{Q}} \mathbb{Q}^{\mathrm{rc}}))$ admit a right exponentiation?

Theorem 2.8. *Every real closed field of formal power series admits a right exponential function, namely*

$$\exp : \mathbb{R}((G^{>0})) \xrightarrow{\sim} 1 + \mathbb{R}((G^{>0})),\, \varepsilon \mapsto \sum_i \frac{\varepsilon^i}{i!}$$

(recall Neumann's lemma, see chapter II)

Proposition 2.9. *Let K be a real closed field and assume that K admits a middle exponential. Then $\overline{K}$ is an exponential Archimedean field.*

Proof. Note that

$$(\overline{K}, +, 0, <) \cong \mathbb{A}' \cong \mathbb{B}' \cong (\overline{K}^{>0}, 1, \cdot, <),$$

therefore f_M is an exponential on $\overline{K}$. $\square$

$\mathbb{K}$ does not admit a middle exponential (e is transcendental, $\mathbb{Q}^{\mathrm{rc}}$ is not an Archimedean exponential field).

Example 2.10. Let E be a countable real closed exponentially closed subfield of $\mathbb{R}$. Note that such an E exists, it can be constructed by induction from $\mathbb{Q}$ by countable iteration of taking real closure, exponential closure and closure under logarithm for positive elements.

Consider $G = \bigcup_{\mathbb{Q}} E$, $\mathbb{K} = E((G))$. Then $\mathbb{K}$ admits a middle and right exponential, but still no left exponential.

Open Question: Does every non-Archimedean real closed field admit a right exponential function?

Theorem 2.11. *(Ron Brown)*
Let (V, v) be a countable dimensional valued vector space. Then V admits a valuation basis.
In particular, if (V_1, v_1) and (V_2, v_2) are countable dimensional valued vector spaces with same skeleton $S(V_1) = S(V_2)$, then they are isomorphic as valued vector spaces, i.e. $(V_1, v_1) \cong (V_2, v_2)$.

Proof. Follows by induction from the following lemma □

Lemma 2.12. *Let V be a valued vector space, W a finite dimensional subspace with valuation basis $\mathcal{B}$ and let $a \in V$. Then $\mathcal{B}$ can be extended to a valuation basis of $< W, a >$.*

Proof. Consider $\{v(b) : b \in \mathcal{B}\}$ finite. So there exists some $a_0 \in W$ such that $v(a - a_0) \notin v(W)$ or, if this is not possible, such that $v(a - a_0) \in v(W)$ is maximal. Without loss of generality, $a \notin W$. If $v(a - a_0) \notin v(W)$, then $\mathcal{B} \cup \{a - a_0\}$ is the required valuation basis of $< W, a >$.
Otherwise set $\gamma := v(a - a_0) \in v(W)$. By the characterization of valuation basis (see chapter I) B_γ forms a basis of $\mathcal{B}(W, \gamma)$. If $\pi(\gamma, a - a_0)$ would live in $\mathcal{B}(W, \gamma)$, there would be a linear combination a_1 of elements of $\mathcal{B}$ with value γ such that $\pi(\gamma, a - a_0 - a_1) = 0$. But this means that $v(a - a_0 - a_1) > \gamma$, a contradiction. So $\pi(\gamma, a - a_0) \notin \mathcal{B}(W, \gamma)$, so $\mathcal{B} \cup \{a - a_0\}$ is valuation independent.

□

Corollary 2.13. *(Answer to the open question in the countable case)*
Let K be a countable non-Archimedean real closed field. Then K admits right exponentiation.

Proof. It can be shown that for any ordered field $S(I_v) \cong S(1 + I_v)$. In particular, by Brown's theorem, if K is countable, I_v and $I_v + 1$ are both countable and have the same skeleton, so they are isomorphic. □