REAL ALGEBRAIC GEOMETRY LECTURE NOTES (20: 25/06/15)

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Contents

1. Principal final segments

Last lecture we studied the order type of the chain Γ^{fs} of non-empty final segments of the set $\Gamma =: v_G(G\backslash\{0\})$.

Lemma 1.1. *The order type of the chain Γ^{fs} is uniquely determined by the order type of Γ, i.e. if Γ_1 and Γ_2 are chains such that $\phi : \Gamma_1 \cong \Gamma_2$ as ordered sets, then $\Gamma_1^{fs} \cong \Gamma_2^{fs}$ as ordered sets.*

Proof. Define $\phi^{\mathrm{fs}} : \Gamma_1^{\mathrm{fs}} \to \Gamma_2^{\mathrm{fs}}$, $F \mapsto \phi(F)$. Verify (ÜA) that ϕ is injective, i.e.

$$F_1 \subsetneq F_2 \Rightarrow \phi(F_1) \subsetneq \phi(F_2) \quad (*)$$

so the map ϕ^{fs} is injective. Also if $F' \in \Gamma_2^{\mathrm{fs}}$, then $\phi^{-1}(F') \in \Gamma_1^{\mathrm{fs}}$, so $\phi^{\mathrm{fs}}(\phi^{-1}(F')) = F'$, showing ϕ^{fs} is surjective.
Finally ϕ^{fs} is order preserving because of $(*)$. □

Recall we noted last lecture that Γ^* is order isomorphic to the totally ordered (inclusion) set of principal final segments, given by

$$\Gamma^* \to \Gamma^{\mathrm{pfs}}, \gamma \mapsto [\gamma, \infty) = \Gamma^+.$$

2. Principal convex subgroups

Definition 2.1. Let G be a totally ordered abelian group and $G_w \neq \{0\}$ a convex subgroup. G_w is said to be a **principal convex subgroup**, if $\exists g \in G$ such that G_w is the smallest convex subgroup containing g.

Remark 2.2. In the notation introduced and used in chapter I, $G_w = C_g$, i.e. G_w is the convex subgroup generated by g.

Lemma 2.3. *Let $G_w \neq \{0\}$ be a convex subgroup of G. The following are equivalent:*

(i) *G_w is principle convex generated by g,*

(ii) *$v_G(G_w) = \Gamma_w$ is a principal final segment, namely $v_G(g)^+$.*

Proof. First show $(i) \Rightarrow (ii)$. Note that $[g] \cap G_w^{\leqslant 0}$ is an initial segment in G_w and that $[g] \cap G_w^{>0}$ is a final segment in G_w. So $v_G(g)$ is the smallest element of the final segment $v_G(G_w)$, i.e. $v_G(G_w) = v_G(g)^+$.
Show now $(ii) \Rightarrow (i)$. Assume that $\exists g \in G^{>0}$ such that $v_G(G_w) = v_G(g)^+$ and argue reversing implications that we must have $[g] \cap G_w^{>0}$ is a final segment, i.e. G_w is the smallest convex subgroup containing g and multiples. $\square$

3. Principal convex rank

Definition 3.1. Let G be a totally ordered abelian group. The **principal rank of** G is the order type of the chain of principle convex subgroups $\neq \{0\}$ of G.

Corollary 3.2. *(Characterization of the principal rank)*
The map $G_w \mapsto \min v_G(G_w)$ is an order reversing bijection between the principal rank of G and Γ. Therefore the principle rank of G is order isomorphic to Γ^{pfs}, i.e. to Γ^*.

Remark 3.3.

- Going down for the rank:

  rank of $K \to$ rank of $G = v(K) \to \Gamma^{\mathrm{fs}}$, $\Gamma = v_G(G) = v_G(v(K))$.

- Going up for the principal rank:

  $\Gamma^{\mathrm{pfs}} \cong \Gamma^* \to$ principal rank of $G \to$ principal rank of K?

Definition 3.4.

(i) A convex subring $K_w \supsetneq K_v$ is said to be a **principal convex subring** if $\exists a \in K^{>0} \backslash K_v$ such that K_w is the smallest convex subring containing a. We say K_w is the convex subring generated by a.

(ii) The **principal rank of** K is the (order type of the) set of all principal convex valuation rings ordered by inclusion. Denote it by $\mathcal{R}^{\mathrm{pr}} \subset \mathcal{R}$.

Definition 3.5. Let $a, b \in K^{>0} \backslash K$. We define a relation

$$a \sim b :\Leftrightarrow \exists n \in \mathbb{N} \text{ such that } a^n \geqslant b \text{ and } b^n \geqslant a.$$

ÜA: Show that $\sim$ is an (Archimedean) equivalence relation. Moreover,

$$a \sim b \Leftrightarrow v(a) \sim^+ v(b) \Leftrightarrow v_G(v(a)) = v_G(v(b)).$$

Therefore

$$K \to v(K) \to v_G(v(K^*)).$$

Theorem 3.6. *(Characterization of the principal rank of an ordered field) For $K_w \in \mathcal{R}$, the following are equivalent:*

(i) *$K_w \in \mathcal{R}^{pr}$ (generated by a)*

(ii) *G_w is a principal convex subgroup of $G = v(K^*)$ generated by $v(a)$.*

(iii) *Γ_w is a principal final segment generated by $v_G(v(a))^+$.*

Corollary 3.7. $\mathcal{R}^{pr} \cong \Gamma^*$.

Corollary 3.8. *Given a chain $\Delta \neq \emptyset$, there exists a non-Archimedean real closed field K such that $\mathcal{R}_K^{\mathrm{pr}}$ is isomorphic to Δ.*

Proof. Take $k = \mathbb{R}$ (for example). Set $\Gamma = \Delta^*$, set $G := \bigoplus_\Gamma \mathbb{R}$ the Hahn sum. Then G has principal rank $\Gamma^* = \Delta^{**} = \Delta$. Set

$$\mathbb{K} = k((G)) = \mathbb{R}((G)).$$

Then $\mathbb{K}$ has principal rank Δ. □