REAL ALGEBRAIC GEOMETRY LECTURE NOTES (13: 28/05/15)

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Contents

1. The field of generalized power series

In order to prove that $k((G))$ is a field, we have seen that it suffices to find a multiplicative inverse for $f \in k((G))$ of the form $f = 1 + s$, where $v(s) > 0$, i.e. support $s \subset G^{>0}$. We already constructed $(1 + s)^{-1}$ via the expansion which gives a summable series by Neumann's lemma.

Today we give an alternative proof by S. Prieß-Crampe, which capitalizes on the fact that $k((G))$ is pseudo-complete.

Proof. Let $v := v_{\min}$ be the canonical valuation on the Hahn product $k((G))$; that is $v(f) = \min \text{support} f$ for $f \neq 0,\ f \in k((G))$. It is enough, as noted, to find an inverse for $f = 1 + s,\ s \neq 0$ with $v(s) > 0$. Note that $v(f) = 0$ and $f(0) = 1$. Denote $\mathbb{K} := k((G))$ and consider the set

$$\Sigma := \{v(1 - fy) : y \in \mathbb{K} \text{ and } 1 - fy \neq 0\}.$$

Note that $\Sigma \neq \emptyset$.

Case 1: Σ has a largest element α. Let $\tilde{y} \in \mathbb{K}$ be such that $v(1-f\tilde{y}) = \alpha$. Set $z := 1 - f\tilde{y}$ and $\hat{y} := \tilde{y} + z(\alpha)t^{\alpha}$. Compute

$$\begin{aligned} v(1 - f\hat{y}) &= v(1 - f\tilde{y} - fz(\alpha)t^{\alpha}) \\ &\geqslant \min\{v(1 - f\tilde{y}), v(fz(\alpha)t^{\alpha})\} = \alpha. \end{aligned}$$

On the other hand

$$\begin{aligned} (1 - f\hat{y})(\alpha) &= (1 - f\tilde{y})(\alpha) - (fz(\alpha)t^{\alpha})(\alpha) \\ &= z(\alpha) - z(\alpha) \\ &= 0. \end{aligned}$$

Thus $v(1 - f\hat{y}) > \alpha$, a contradiction to the maximal choice of α, unless $1 - f\hat{y} = 0$, so $1 = f\hat{y}$ and therefore $\hat{y} = f^{-1}$.

(**Recall:** In chapter 1 we have shown that $\mathbb{K}$ is pseudo-complete, or equivalently, maximally valued).

Case 2: Σ has no largest element. Thus, there is a strictly increasing sequence $\{\pi_\rho\}_{\rho<\sigma}$ of Σ where σ is a limit ordinal and $\{\pi_\rho\}_{\rho<\sigma}$ is cofinal in Σ.
For every $\rho < \sigma$ choose $y_\rho \in \mathbb{K}$ such that $v(1-fy_\rho) = \pi_\rho$. Now for $\mu < \nu < \sigma$ we have $\pi_\mu < \pi_\nu$. We claim that $\{y_\rho\}_{\rho<\sigma}$ is pseudo-Cauchy. Indeed

$$\begin{aligned} v(y_\mu - y_\nu) &= v(1 - fy_\mu + 1 - fy_\nu) \\ &= \min\{\pi_\mu, \pi_\nu\} = \pi_\nu. \end{aligned}$$

So the sequence is indeed pseudo-Cauchy. Now since $\mathbb{K}$ is pseudo-complete let y^* be a pseudo-limit of $\{y_\rho\}_{\rho<\sigma}$, i.e. $v(y^* - y_\rho) = \pi_\rho$ for all $\rho < \sigma$. Assume that $1 - fy^* \neq 0$. Then $\tau := v(1 - fy^*) \in \Sigma$. By cofinality of $\{\pi_\rho\}_{\rho<\sigma}$ there is a ρ large enough such that $\tau < \pi_\rho$. On the other hand

$$\begin{aligned} \tau = v(1 - fy^*) &= v(1 - fy_\rho + fy_\rho - fy^*) \\ &\geqslant \min\{v(1 - fy_\rho), v(fy_\rho - fy^*)\} \\ &\geqslant \pi_\rho, \end{aligned}$$

a contradiction.

□

Remark 1.1.

(*i*) We have used the fact that for $0 \neq s, r \in \mathbb{K}$, we have

$$v_{\min}(sr) = v_{\min}(s) + v_{\min}(r).$$

This follows immediately from the definition of multiplication of series in the convolution product.

(*ii*) Note that here the pseudo-limit y^* turns out to be unique. We can conclude that the breadth of $\{\pi_\rho\}_{\rho<\sigma}$ is $\{0\}$.

In conclusion, for $k \subseteq \mathbb{R}$ an Archimedean field and G any non-trivial ordered abelian group, the field $\mathbb{K} = k((G))$ endowed with $<_{\text{lex}}$ is a totally ordered non-Archimedean field. Its natural valuation is $v_{\min}$, its value group is G and its residue field k. Note that in general $k((G))$ needs not to be a real closed field.

In the next chapter we will give necessary and sufficient conditions on k and G such that $\mathbb{K} = k((G))$ is a real closed field.

2. Hardy fields

Definition 2.1. Consider the set of all real valued functions defined on positive real half lines:

$$\mathcal{F} := \{f \mid f\colon [a,\infty) \to \mathbb{R} \text{ or } f\colon (a,\infty) \to \mathbb{R},\ a \in \mathbb{R}\} \cup \{-\infty\}.$$

Define an equivalence relation on $\mathcal{F}$ by

$$f \sim g \;\Leftrightarrow\; \exists\, N \in \mathbb{N} \text{ s.t. } f(x) = g(x)\; \forall\, x \geqslant N.$$

Let $[f]$ denote the equivalence class of f, also called the "germ of f at ∞". We identify $f \in \mathcal{F}$ with its germ $[f]$.

We denote by $\mathcal{G} := \mathcal{F}/\sim$ the set of all germs. Note that $\mathcal{G}$ is a commutative ring with 1 by defining

$$[f] + [g] := [f + g]$$
$$[f] \cdot [g] := [f \cdot g]$$

Note that $\mathcal{G}$ is not a field. For example $[\sin x]$ is not invertible.

Definition 2.2. A subring H of $\mathcal{G}$ is a **Hardy field** if it is a field with respect to the operations above and if it is closed under differentiation of germs, i.e. $\forall f \in H : f' \in H$ exists and is well-defined ultimately (i.e. for all $x > N \in \mathbb{N}$).

Remark 2.3. (defininig a total order on a Hardy field).
Let H be a Hardy field and $f \in H$, $f \neq 0$. Since $1/f \in H$, $f(x) \neq 0$ ultimately. Moreover since $f' \in H$, f is ultimately differentiable and thus ultimately continuous. Therefore, by the Intermediate Value Theorem, the sign of f is ultimately constant and non-zero (i.e. f is strictly positive on some interval (N, ∞) or f is strictly negative on some interval (N, ∞)).
Thus we can define

$$f > 0 \text{ if ult sign } f = 1,$$

respectively

$$f < 0 \text{ if ult sign } f = -1.$$

Verify that $(H, <)$ is a totally ordered field.