REAL ALGEBRAIC GEOMETRY LECTURE NOTES (03: 20/04/15)

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Contents

1. Hahn Sandwich Proposition

Lemma 1.1.

(*i*) $\coprod_{\gamma\in\Gamma} B(\gamma) \subseteq \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$.

(*ii*)

$$\begin{aligned} S(\coprod_{\gamma\in\Gamma} B(\gamma)) &\cong [\,\Gamma, \{B(\gamma) : \gamma \in \Gamma\}\,] \\ &\cong S(\mathrm{H}_{\gamma\in\Gamma} B(\gamma)). \end{aligned}$$

We shall show that if $Z = Q$ is a field and (V, v) is a valued Q-vector space with skeleton $S(V) = [\,\Gamma, \{B(\gamma) : \gamma \in \Gamma\,]$, then

$$\left(\coprod_{\gamma\in\Gamma} B(\gamma), v_{\min}\right) \hookrightarrow (V, v) \hookrightarrow (\mathrm{H}_{\gamma\in\Gamma} B(\gamma), v_{\min}).$$

2. Immediate extensions

Definition 2.1. Let (V_i, v_i) be valued Q-vector spaces $(i = 1, 2)$.

(1) Let $V_1 \subseteq V_2$ be a Q-subspace with $v_1(V_1) \subseteq v_2(V_2)$. We say that (V_2, v_2) is an **extension** of (V_1, v_1), and we write

$$(V_1, v_1) \subseteq (V_2, v_2),$$

if ${v_2}_{|_{V_1}} = v_1$.

(2) If $(V_1, v_1) \subseteq (V_2, v_2)$ and $\gamma \in v_1(V_1)$, the map

$$\begin{array}{rcl} B_1(\gamma) & \longrightarrow & B_2(\gamma) \\ x + (V_1)_\gamma & \mapsto & x + (V_2)_\gamma \end{array}$$

is a natural identification of $B_1(\gamma)$ as a Q-subspace of $B_2(\gamma)$. The extension $(V_1, v_1) \subseteq (V_2, v_2)$ is **immediate** if $\Gamma := v_1(V_1) = v_2(V_2)$ and $\forall \gamma \in v_1(V_1)$

$$B_1(\gamma) = B_2(\gamma).$$

Equivalently, $(V_1, v_1) \subseteq (V_2, v_2)$ is immediate if $S(V_1) = S(V_2)$.

Lemma 2.2. *(Characterization of immediate extensions)*
The extension $(V_1, v_1) \subseteq (V_2, v_2)$ *is immediate if and only if*

$$\forall x \in V_2,\ x \neq 0,\ \exists y \in V_1 \ \text{ such that } \ v_2(x - y) > v_2(x).$$

Proof. We show that in a valued Q-vector space (V, v), for every $x, y \in V$

$$v(x-y) > v(x) \iff \begin{cases} (i) \quad \gamma = v(x) = v(y) \ \text{ and} \\ (ii) \quad \pi(\gamma, x) = \pi(\gamma, y). \end{cases}$$

($\Leftarrow$) Suppose (i) and (ii). So $x, y \in V^\gamma$ and $x - y \in V_\gamma$.
Then $v(x - y) > \gamma = v(x)$.

($\Rightarrow$) Suppose $v(x - y) > v(x)$. We show (i) and (ii).
Assume for a contradiction that $v(x) \neq v(y)$. Then $v(x - y) = \min\{v(x), v(y)\}$. So if $v(x) > v(y)$, then $v(y) = v(x - y) > v(x)$ and if $v(y) > v(x)$, then $v(x) = v(x - y) > v(x)$. Both is obviously a contradiction. Thus, $v(x) = v(y)$. (ii) is analogue.

□

Example 2.3. $(\bigsqcup_{\gamma\in\Gamma} B(\gamma),\ v_{\min}) \ \subseteq \ (\mathrm{H}_{\gamma\in\Gamma} B(\gamma),\ v_{\min})$
is an immediate extension.

Proof. Given $x \in \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$, $x \neq 0$, set

$$\gamma_0 := \min \operatorname{support}(x) \qquad \text{and} \qquad x(\gamma_0) := b_0 \in B(\gamma_0).$$

Let $y \in \bigsqcup_{\gamma\in\Gamma} B(\gamma)$ such that

$$y(\gamma) = \begin{cases} 0 & \text{if } \gamma \neq \gamma_0 \\ b_0 & \text{if } \gamma = \gamma_0. \end{cases}$$

Namely $y = b_0 \chi_{\gamma_0}$, where

$$\chi_{\gamma_0} \colon \Gamma \longrightarrow Q$$

$$\chi_{\gamma_0}(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_0 \\ 0 & \text{if } \gamma \neq \gamma_0. \end{cases}$$

Then $v_{\min}(x - y) > \gamma_0 = v_{\min}(x)$ (because $(x - y)(\gamma_0) = x(\gamma_0) - y(\gamma_0) = b_0 - b_0 = 0$).

□

3. Valuation independence

Definition 3.1. $\mathcal{B} = \{x_i : i \in I\} \subseteq V \setminus \{0\}$ is **Q-valuation independent** if for $q_i \in Q$ with $q_i = 0$ for all but finitely many $i \in I$, we have

$$v\left(\sum_{i\in I} q_i x_i\right) = \min_{i\in I, q_i \neq 0}\{v(x_i)\}.$$

Remark 3.2.

(1) Q-linear independence $\not\Rightarrow$ Q-valuation independence.
Consider $(\coprod_2 \mathbb{Q}, v_{\min})$ and the elements $x_1 = (1,1),\ x_2 = (1,0)$.

(2) $\mathcal{B} \subseteq V\setminus\{0\}$ is Q-valuation independent $\Rightarrow$ $\mathcal{B}$ is Q-linear independent.
Else $\exists\ q_i \neq 0$ with $\sum q_i x_i = 0$ and $\min\{v(x_i)\} = v(\sum q_i x_i) = \infty$, a contradiction.

Proposition 3.3. *(Characterization of valuation independence)*
Let $\mathcal{B} \subseteq V \setminus \{0\}$. Then $\mathcal{B}$ is Q-valuation independent if and only if $\forall\, n \in \mathbb{N}$ and $\forall\, b_1, \ldots, b_n \in \mathcal{B}$ pairwise distinct with $v(b_1) = \cdots = v(b_n) = \gamma$, the coefficients

$$\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n) \in B(\gamma)$$

are Q-linear independent in the Q-vector space $B(\gamma)$.

Proof.

($\Rightarrow$) Let $b_1, \ldots, b_n \in \mathcal{B}$ with $v(b_1) = \cdots = v(b_n) = \gamma$ and suppose for a contradiction that

$$\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n) \in B(\gamma)$$

are not Q-linear independent. So there are $q_1, \ldots, q_n \in Q$ non-zero such that $\pi(\gamma, \sum q_i b_i) = 0$, so $v(\sum q_i b_i) > \gamma$. This contradicts the valuation independence.

($\Leftarrow$) We show that

$$v\left(\sum q_i b_i\right) = \min\{v(b_i)\} = \gamma.$$

Since $\pi(\gamma, b_1), \ldots, \pi(\gamma, b_n)$ are Q-linear independent in $B(\gamma)$,

$$\pi\left(\gamma, \sum q_i b_i\right) \neq 0,$$

i.e. $v(\sum q_i b_i) \leqslant \gamma$.
On the other hand $v(\sum q_i b_i) \geqslant \gamma$, so $v(\sum q_i b_i) = \gamma = \min\{v(b_i)\}$.

□

4. Maximal valuation independence

By Zorn's lemma, maximal valuation independent sets exist:

Corollary 4.1. *(Characterization of maximal valuation independent sets)* $\mathcal{B} \subseteq V \setminus \{0\}$ *is maximal valuation independent if and only if* $\forall\, \gamma \in v(V)$

$$\mathcal{B}_\gamma := \{\pi(\gamma, b) : b \in \mathcal{B}, v(b) = \gamma\}$$

is a Q*-vector space basis of* $B(\gamma)$*.*

Corollary 4.2. *Let* $\mathcal{B} \subseteq V \setminus \{0\}$ *be valuation independent in* (V, v)*. Then* $\mathcal{B}$ *is maximal valuation independent if and only if the extension*

$$\langle\, \mathcal{B}\,\rangle := (V_0, v_{|V_0}) \ \subseteq\ (V, v)$$

is an immediate extension.

Proof.

($\Rightarrow$) Assume $\mathcal{B} \subseteq V$ is maximal valuation independent. We show $V_0 \subseteq V$ is immediate.

If not $\exists\, x \in V$, $x \neq 0$, such that

$$\forall\, y \in V_0 :\ v(x - y) \leqslant v(x).$$

We will show that in this case $\mathcal{B} \cup \{x\}$ is valuation independent (which will contradict our maximality assumption). Consider $v(y_0 + qx)$, $q \in Q$, $q \neq 0$, $y_0 \in V_0$. Set $y := -y_0/q$. We claim that

$$v(y_0 + qx) = v(x - y) = \min\{v(x), v(y)\} = \min\{v(x), v(y_0)\}.$$

This follows immediately from

Fact: $v(x - y) \leqslant v(x) \iff v(x - y) = \min\{v(x), v(y)\}$.

Proof of the fact. The implication ($\Leftarrow$) is trivial. To see ($\Rightarrow$), assume that $v(x - y) > \min\{v(x), v(y)\}$.

If $\min\{v(x), v(y)\} = v(x)$, then we have the contradiction

$$v(x) \geqslant v(x - y) > \min\{v(x), v(y)\} = v(x).$$

If $\min\{v(x), v(y)\} = v(y) < v(x)$, then $v(y) = v(x - y) > v(y)$, again a contradiction.

($\Leftarrow$) Now assume that $(V_0, v_{|V_0}) \subseteq (V, v)$ is immediate. We show that $\mathcal{B}$ is maximal valuation independent.

If not, $\mathcal{B} \cup \{x\}$ is valuation independent for some $x \in V \setminus \{0\}$ with $x \notin \mathcal{B}$. So $\forall y \in V_0$ we get $v(x - y) \leqslant v(x)$ by the fact above. This contradicts that $(V_0, v_{|V_0}) \subseteq (V, v)$ is immediate.

□