

Universität Konstanz

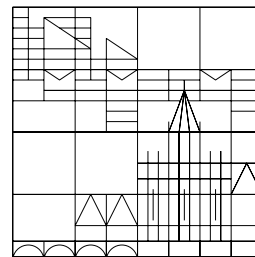
Fachbereich

Mathematik und Statistik

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Real Algebraic Geometry II

Exercise Sheet 2

Exercise 1 (Cantor's Theorem)

Let (A, \leq) be a countable dense linear order without endpoints.

Show that $(A, \leq) \cong (\mathbb{Q}, \leq)$.

Hint: Define for every $i \in \mathbb{N}$ sets $A_i \subseteq A$ and $B_i \subseteq \mathbb{Q}$ such that $A_{i-1} \subseteq A_i$, $B_{i-1} \subseteq B_i$, $\bigcup_{i=1}^{\infty} A_i = A$ and $\bigcup_{i=1}^{\infty} B_i = \mathbb{Q}$. Further define for every $i \in \mathbb{N}$ $f_i : A_i \rightarrow B_i$ and then use this to define an isomorphism between (A, \leq) and (\mathbb{Q}, \leq) .

If i is even first define A_i and then choose B_i accordingly and if i is odd first define B_i and then choose A_i accordingly.

Exercise 2

Let (A, \leq_A) be a countable dense linear order without endpoints. Let (B, \leq_B) be an arbitrary countable linear order.

Show that (B, \leq_B) is isomorphic to a subordering of (A, \leq_A) .

In particular every countable ordinal embeds into (\mathbb{Q}, \leq) .

Exercise 3

Let (A, \leq) be a linear order. Assume there exists $(B, \leq) \subseteq (A, \leq)$ with (B, \leq) countable and dense in (A, \leq) . Let $(C, \leq) \subseteq (A, \leq)$ be well-ordered.

Show that C is countable.

In particular every well-ordered subset of (\mathbb{R}, \leq) is countable.

Hint: Let μ denote the order type of C . Fix an ordinal enumeration $C = \{c_\alpha \mid \alpha \in \mu\}$ with $c_\alpha < c_\beta$ for all $\alpha < \beta$.

Reminder:

Let (A, \leq) be a linear order.

- (A, \leq) is called dense, if for all $a, b \in A$ with $a < b$ there exists $x \in A$ such that $a < x < b$.
- (A, \leq) is an order without endpoints, if for all $a \in A$ there exist $x, y \in A$ such that $x < a < y$.
- $(B, \leq) \subseteq (A, \leq)$. Then (B, \leq) is called dense in (A, \leq) , if for all $a, b \in A$ with $a < b$ there exists $x \in B$ such that $a < x < b$.

A set A is called countable if there exists an injective map from A into \mathbb{N} .

The exercise will be collected **Thursday, 30/04/2015** until 10.00 at the box near F 441.

<http://www.math.uni-konstanz.de/~dupont/rag.htm>