Universität Konstanz

Fachbereich
Mathematik und Statistik

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Real Algebraic Geometry II

Exercise Sheet 2

Exercise 1 (Cantor's Theorem)
Let $(A, \leq)$ be a countable dense linear order without endpoints.
Show that $(A, \leq) \cong (\mathbb{Q}, \leq)$.
Hint: Define for every $i \in \mathbb{N}$ sets $A_i \subseteq A$ and $B_i \subseteq \mathbb{Q}$ such that $A_{i-1} \subseteq A_i$, $B_{i-1} \subseteq B_i$, $\bigcup_{i=1}^{\infty} A_i = A$ and $\bigcup_{i=1}^{\infty} B_i = \mathbb{Q}$. Further define for every $i \in \mathbb{N}$ $f_i : A_i \to B_i$ and then use this to define an isomorphism between $(A, \leq)$ and $(\mathbb{Q}, \leq)$.
If i is even first define A_i and then choose B_i accordingly and if i is odd first define B_i and then choose A_i accordingly.

Exercise 2
Let $(A, \leq_A)$ be a countable dense linear order without endpoints. Let $(B, \leq_B)$ be an arbitrary countable linear order.
Show that $(B, \leq_B)$ is isomorphic to a subordering of $(A, \leq_A)$.
In particular every countable ordinal embeds into $(\mathbb{Q}, \leq)$.

Exercise 3
Let $(A, \leq)$ be a linear order. Assume there exists $(B, \leq) \subseteq (A, \leq)$ with $(B, \leq)$ countable and dense in $(A, \leq)$. Let $(C, \leq) \subseteq (A, \leq)$ be well-ordered.
Show that C is countable.
In particular every well-ordered subset of $(\mathbb{R}, \leq)$ is countable.
Hint: Let μ denote the order type of C. Fix an ordinal enumeration
$C = \{c_\alpha \mid \alpha \in \mu\}$ with $c_\alpha < c_\beta$ for all $\alpha < \beta$.

Reminder:
Let $(A, \leq)$ be a linear order.

- $(A, \leq)$ is called dense, if for all $a, b \in A$ with $a < b$ there exists $x \in A$ such that $a < x < b$.
- $(A, \leq)$ is an order without endpoints, if for all $a \in A$ there exist $x, y \in A$ such that $x < a < y$.
- $(B, \leq) \subseteq (A, \leq)$. Then $(B, \leq)$ is called dense in $(A, \leq)$, if for all $a, b \in A$ with $a < b$ there exists $x \in B$ such that $a < x < b$.

A set A is called countable of there exists an injective map from A into $\mathbb{N}$.

The exercise will be collected **Thursday, 30/04/2015** until 10.00 at the box near F 441.
