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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 08

These exercises will be collected Tuesday 29th June in the mailbox n.14 of the Mathematics department.

Notation: We denote by $\overline{A}$ the topological closure of a set A.

Definition. Let X be a topological space and $A, B \subseteq X$ two disjoint subsets. We say that:

- A and B are **separated** if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$.
- A and B are **separated by neighbourhoods** if there are open disjoint $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.
- A and B are **separated by a function** if there is a continuous function $f \colon X \to [0,1] \subset \mathbb{R}$ such that $f(a) = 0\ \forall\, a \in A$ and $f(b) = 1\ \forall\, b \in B$.

Let X be a topological space. We recall some separation axioms:

- X is $\mathbf{T_1}$ if any two distinct points of X are separated.
- X is $\mathbf{T_2}$ or **Hausdorff** if any two distinct points of X are separated by neighbourhoods.
- We say that X **has property** $(*)$ if

$(*)$ any two disjoint closed subsets of X are separated by neighbourhoods.

In the literature, sometimes spaces with property $(*)$ are called T_4, and in this case a space which is T_1 and T_4 is said to be normal.

Other times a space is called normal if it has property $(*)$ and T_4 if in addition it is T_1.

For this reason we will just say that X has property $(*)$ with no additional name.

The aim of the first exercise is to show that a topological space X has property $(*)$ if and only if every pair of closed disjoint sets is separated by a function.

In order to apply this result to the proof of Haviland's Theorem we also need exercise $3(i)$.

1. Let X be a topological space.

(i) (**Urysohn's Lemma**) Suppose X has property $(*)$. Show that any two closed disjoint sets are separated by a function in the following way:

(a) Let $A, B \subseteq X$ closed disjoint. Set

$$R = \{r = k/2^n \in \mathbb{Q} : 0 \leqslant r \leqslant 1,\ k, n \in \mathbb{Z}_+\}.$$

Show that for every $r \in R$ there is an open set $U(r)$ such that:

- $A \subseteq U(r)$,
- $U(r) \cap B = \emptyset$,
- $r < r' \ \Rightarrow\ \overline{U(r)} \subseteq U(r')$.

(*Hint*: Prove it by induction on $n =$ the exponent of the dyatic fractions)

(b) Replace $U(1)$ by X and define

$$f(y) := \inf\{r \in R : y \in U(r)\}.$$

Show that A and B are separated by f, namely $f\colon X \to [0,1] \subset \mathbb{R}$ is continuous, $f(a) = 0\ \forall\, a \in A$ and $f(b) = 1\ \forall\, b \in B$.

(ii) Show that if two disjoint closed subset of X are separated by a function, then they are separated by neighbourhoods.

2. Let $X = [-1,1]^2 \subset \mathbb{R}^2$ with the topology induced by the Euclidean topology on $\mathbb{R}^2$. Consider the function

$$\begin{aligned} g\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto -x^2 - 1/10 \end{aligned}$$

Set

$$A = \text{Graph}(g) \cap X \qquad \text{and} \qquad B = \{(1,1)\}.$$

Find a family of open sets $U(r)$ in X as in exercise 1(i) and define a function separating A and B accordingly.

3.

(i) Let X be a compact Hausdorff topological space. Show that X has property $(*)$.

(ii) Is also the converse true?

$$X \text{ has property } (*) \overset{?}{\Longrightarrow} X \text{ compact}$$

$$X \text{ has property } (*) \overset{?}{\Longrightarrow} X \text{ Hausdorff}$$

If the answers are positive, prove them. If negative, provide counterexamples.