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SS2010

ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 03

These exercises will be collected Tuesday 11 in the mailbox n.14 of the Mathematics department.

1. Let $A = \mathbb{R}[\underline{\mathrm{x}}]$, S a finite subset of A, $T = T_S$ the preordering of A generated by S, $\mathrm{Sper}_T(A) := \{P : P \text{ is an ordering and } A \supset P \supset T\}$ and $K = K_S$ the basic closed semialgebraic subset of $\mathbb{R}^n$ associated to S. Consider the following map:

$$\begin{array}{rcl} P\colon K & \longrightarrow & \mathrm{Sper}_T(A) \\ \underline{x} & \mapsto & P_{\underline{x}} := \{f \in A : f(\underline{x}) \geqslant 0\}. \end{array}$$

Show that P is well-defined and $P(K)$ is dense in $\mathrm{Sper}_T(A)$ with respect to the constructible topology.

2. Let f be a homogeneous polynomial in $\mathbb{R}[\underline{x}]$. Show that if f is sum of squares then every sum of square representation of f consists of homogeneous polynomials, namely:

$$f = f_1^2 + \cdots + f_k^2 \;\Rightarrow\; f_i \text{ is homogeneous } \forall\, i = 1, \ldots, k.$$

3. Show that:

(a) every convex polytope in $\mathbb{R}^k$ is closed and bounded (so compact) in $\mathbb{R}^k$ with respect to the Euclidean topology;

(b) every convex polytope is the convex hull of its vertices;

(c) any vertex of a convex polytope is an extremal point.

4. A subset $\mathcal{C}$ of $\mathbb{R}^n$ is a **convex cone** if it is closed under addition and under multiplication by non-negative scalars, i.e.:

$$\underline{x},\, \underline{y} \in \mathcal{C} \;\Rightarrow\; \underline{x} + \underline{y} \in \mathcal{C}$$
$$\underline{x} \in \mathcal{C},\, \lambda \geqslant 0 \;\Rightarrow\; \lambda \underline{x} \in \mathcal{C}.$$

(i) Show that a subset of $\mathbb{R}^n$ is a convex cone if and only if it contains all the non-negative linear combinations of its elements.

For $S \subseteq R^n$, we denote by **cone**(S) the set of all non-negative linear combinations of elements from S and we call it **the convex cone generated by** S.

Show that:

(ii) for every $S \subseteq \mathbb{R}^n$, cone(S) is smallest convex cone containing S;

(iii) if $S \subseteq \mathbb{R}^n$ is convex, then $\text{cone}(S) = \{\lambda \underline{x} : \lambda \geqslant 0,\, \underline{x} \in S\}$.