POSITIVE POLYNOMIALS LECTURE NOTES (06: 29/04/10 - BEARBEITET 10/01/19)

SALMA KUHLMANN

Contents

1. GENERALITIES ABOUT POLYNOMIALS

Definition 1.1. For a **polynomial** $p \in \mathbb{R}[X_1, \dots, X_n]$, we write

$$p(\underline{X}) = \sum_{\underline{i} \in \mathbb{Z}_+^n} c_i \ \underline{X}^{\underline{i}} \ ; \ c_i \in \mathbb{R},$$

where $\underline{X}^{\underline{i}} = X_1^{i_1} \dots X_n^{i_n}$ is a monomial of degree $= |\underline{i}| = \sum_{k=1}^{n} i_k$ and $c_i \ \underline{X}^{\underline{i}}$ is a term.

Definition 1.2. A polynomial $p(\underline{X}) \in \mathbb{R}[\underline{X}]$ is called **homogeneous** or **form** if all terms in p have the same degree.

Notation 1.3. $\mathcal{F}_{n,m} := \big\{ F \in \mathbb{R}[X_1, \dots, X_n] \mid F \text{ is a form and } \deg(F) = m \big\}$, the set of all forms in n variables of degree m (also called set of n-ary m-ics forms), for $n, m \in \mathbb{N}$.
Convention: $0 \in \mathcal{F}_{n,m}$.

Definition 1.4. Let $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree m. The **homogenization** of p w.r.t X_{n+1} is defined as

$$p_h(X_1, \dots, X_n, X_{n+1}) := X_{n+1}^m \ p\left(\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right)$$

Note that p_h is a homogeneous polynomial of degree m and in $n+1$ variables i.e. $p_h \in \mathcal{F}_{n+1,m}$.

Proposition 1.5. (1) Let $p(\underline{X}) \in \mathbb{R}[X_1, ...X_n]$, $\deg(p) = m$, then

$$\text{number of monomials of } p \leq \binom{m+n}{n}$$

(2) Let $F(\underline{X}) \in \mathcal{F}_{n,m}$, then

$$\text{number of monomials of } F \leq N := \binom{m+n-1}{n-1}$$ □

Remark 1.6. $\mathcal{F}_{n,m}$ is a finite dimensional real vector space with $\mathcal{F}_{n,m} \simeq \mathbb{R}^N$.

2. PSD- AND SOS- POLYNOMIALS

Definition 2.1. (1) $p(\underline{X}) \in \mathbb{R}[\underline{X}]$ is **positive semidefinite (psd)** if

$$p(\underline{x}) \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n.$$

(2) $p(\underline{X}) \in \mathbb{R}[\underline{X}]$ is **sum of squares (SOS)** if $\exists \ p_i \in \mathbb{R}[\underline{X}]$ s.t.

$$p(\underline{X}) = \sum_i p_i(\underline{X})^2.$$

Notation 2.2. $\mathcal{P}_{n,m}$:= set of all forms $F \in \mathcal{F}_{n,m}$ which are psd, and

$\sum_{n,m}$:= set of all forms $F \in \mathcal{F}_{n,m}$ which are sos.

Lemma 2.3. If a polynomial p is psd then p has even degree. □

Remark 2.4. From now on (using lemma 2.3) we will often write $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$.

Lemma 2.5. Let p be a homogeneous polynomial of degree 2d, and p sos. Then every sos representation of p consists of homogeneous polynomials only, i.e.

$p(\underline{X}) = \sum_i p_i(\underline{X})^2 \Rightarrow p_i(\underline{X})$ homogenous of degree d, i.e. $p_i \in \mathcal{F}_{n,d}$. □

Remark 2.6. The properties of psd-ness and sos-ness are preserved under homogenization (see the following lemma).

Lemma 2.7. Let $p(\underline{X})$ be a polynomial of degree m. Then

(1) p is psd iff p_h is psd,

(2) p is sos iff p_h is sos. □

So we can focus our investigation of psdness of polynomials versus sosness of polynomials to those of forms, i.e. study and compare $\sum_{n,m} \subseteq \mathcal{P}_{n,m}$.

Theorem 2.8. **(Hilbert)** $\sum_{n,m} = \mathcal{P}_{n,m}$ iff

(i) $n = 2$ [i.e. binary forms] or

(ii) $m = 2$ [i.e. quadratic forms] or

(iii) $(n, m) = (3, 4)$ [i.e. ternary quartics].

For the ternary quartics case ($\mathcal{F}_{3,4}$), we shall study the **convex cones** $\mathcal{P}_{n,m}$ and $\sum_{n,m}$.

3. CONVEX SETS, CONES AND EXTREMALITY

Definition 3.1. A subset C of $\mathbb{R}^n$ is **convex set** if $\underline{a}, \underline{b} \in C \Rightarrow \lambda\underline{a}+(1-\lambda)\underline{b} \in C$, for all $0 < \lambda < 1$.

Proposition 3.2. The intersection of an arbitrary collection of convex sets is convex.

Notation 3.3. $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$.

Definition 3.4. Let $\underline{c}_1, \ldots, \underline{c}_k \in \mathbb{R}^n$. A **convex combination** of $\underline{c}_1, \ldots, \underline{c}_k$ is any vector sum

$$\alpha_1\underline{c}_1 + \ldots + \alpha_k\underline{c}_k, \text{ with } \alpha_1, \ldots, \alpha_k \in \mathbb{R}_+ \text{ and } \sum_{i=1}^{k} \alpha_i = 1.$$

Theorem 3.5. A subset $C \subseteq \mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements.

Proof. ($\Leftarrow$) clear
($\Rightarrow$) Let $C \subseteq \mathbb{R}^n$ be a convex set. By definition C is closed under taking convex combinations with two summands. We show that it is also closed under finitely many summands.
Let $k > 2$. By Induction on k, assuming it true for fewer than k.
Given a convex combination $\underline{c} = \alpha_1\underline{c}_1 + \ldots + \alpha_k\underline{c}_k$, with $\underline{c}_1, \ldots, \underline{c}_k \in C$
Note that we may assume $0 < \alpha_i < 1$ for $i = i, \ldots, k$; otherwise we have fewer than k summands and we are done.
Consider $\underline{d} = \dfrac{\alpha_2}{1-\alpha_1}\underline{c}_2 + \ldots + \dfrac{\alpha_k}{1-\alpha_1}\underline{c}_k$
we have $\dfrac{\alpha_2}{1-\alpha_1}, \ldots, \dfrac{\alpha_k}{1-\alpha_1} > 0$ and $\dfrac{\alpha_2}{1-\alpha_1} + \ldots + \dfrac{\alpha_k}{1-\alpha_1} = 1$

Thus $\underline{d}$ is a convex combination of $k-1$ elements of C and $\underline{d} \in C$ by induction. Since $\underline{c} = \alpha_1 \underline{c}_1 + (1-\alpha_1)\underline{d}$, it follows that $\underline{c} \in C$. □

Definition 3.6. The intersection of all convex sets containing a given subset $S \subseteq \mathbb{R}^n$ is called the **convex hull** of S and is denoted by **cvx(S)**.

Remark 3.7. The convex hull of $S \subseteq \mathbb{R}^n$ is a convex set and is the uniquely defined smallest convex set containing S.

Theorem 3.8. For any $S \subseteq \mathbb{R}^n$,
$\mathrm{cvx}(S) =$ the set of all convex combinations of the elements of S.

Proof. ($\supseteq$) The elements of S belong to $\mathrm{cvx}(S)$, so all their convex combinations belong to $\mathrm{cvx}(S)$ by Theorem 3.5.

($\subseteq$) On the other hand we observe that the set of convex combinations of elements of S is itself a convex set:
let $\underline{c} = \alpha_1\underline{c}_1 + \ldots + \alpha_k\underline{c}_k$ and $\underline{d} = \beta_1\underline{d}_1 + \ldots + \beta_l\underline{d}_l$, where $\underline{c}_i, \underline{d}_i \in S$, then
$\lambda\underline{c} + (1-\lambda)\underline{d} = \lambda\alpha_1\underline{c}_1 + \ldots + \lambda\alpha_k\underline{c}_k + (1-\lambda)\beta_1\underline{d}_1 + \ldots + (1-\lambda)\beta_l\underline{d}_l$, $0 \leq \lambda \leq 1$ is
just another convex combination of elements of S.
So by minimality property of $\mathrm{cvx}(S)$, it follows that $\mathrm{cvx}(S) \subseteq$ the set of all convex combinations of the elements of S. □

Corollary 3.9. The convex hull of a finite subset $\{\underline{s}_1, \ldots, \underline{s}_k\} \subseteq \mathbb{R}^n$ consists of all the vectors of the form $\alpha_1\underline{s}_1 + \ldots + \alpha_k\underline{s}_k$ with $\alpha_1, \ldots, \alpha_k \geq 0$ and $\sum_i \alpha_i = 1$. □

Definitions 3.10. (1) A set which is the convex hull of a finite subset of $\mathbb{R}^n$ is called a **convex polytope**, i.e. $C \subseteq \mathbb{R}^n$ is a convex polytope if $C = \mathrm{cvx}(S)$ for some finite $S \subseteq \mathbb{R}^n$.

(2) A point in a polytope is called a **vertex** if it is not on the line segment joining any other two distinct points of the polytope.

Remark 3.11. (1) Convex polytope is necessarily closed and bounded, i.e. compact.

(2) A convex polytope is always the convex hull of its vertices.

More general version for compact sets is the Krein Milman theorem:

Theorem 3.12. **(Krein-Milman)** Let $C \subseteq \mathbb{R}^n$ be a compact and convex set. Then C is the convex hull of its extreme points. □

Definitions 3.13. $\underline{x} \in C$ is **extreme** if $C \setminus \{\underline{x}\}$ is convex.