

REAL ALGEBRAIC GEOMETRY LECTURE NOTES
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1. THE FIELD $\mathbb{R}(x)$

Let us consider again the field $\mathbb{R}(x)$ of the rational functions on $\mathbb{R}[x]$:

Example 1.1. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{R}[x]$ and let $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$ (and therefore actually $f(x) = a_n x^n + \dots + a_k x^k$). We define

$$f(x) > 0 \Leftrightarrow a_k > 0$$

and then for every $f(x), g(x) \in \mathbb{R}[x]$ with $g(x) \neq 0$ we define

$$\frac{f(x)}{g(x)} \geq 0 \Leftrightarrow f(x)g(x) \geq 0.$$

This is a total order on

$$\mathbb{R}(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{R}[x] \text{ and } g(x) \neq 0 \right\}$$

which makes $(\mathbb{R}(x), \leq)$ an ordered field.

Remark 1.2. By the definition above

$$f(x) = x - r < 0 \quad \forall r \in \mathbb{R}, r > 0.$$

Therefore the element $x \in \mathbb{R}(x)$ is such that

$$0 < x < r \quad \forall r \in \mathbb{R}, r > 0.$$

We can see that there is no other ordering on $\mathbb{R}(x)$ which satisfies the above property:

Proposition 1.3. *Let \leq be the ordering on $\mathbb{R}(x)$ defined in 1.1. Then \leq is the unique ordering on $\mathbb{R}(x)$ such that*

$$0 < x < r \quad \forall r \in \mathbb{R}, r > 0.$$

Proof. Assume that \leq is an ordering on $\mathbb{R}(x)$ such that

$$0 < x < r \quad \forall r \in \mathbb{R}, r > 0.$$

Then (see Proposition 2.4 of last lecture)

$$0 < x^m < r \quad \forall m \geq 1, m \in \mathbb{N}, \forall r > 0, r \in \mathbb{R}.$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_k x^k \in \mathbb{R}[x]$ with $k \in \mathbb{N}$ the smallest index such that $a_k \neq 0$. We want to prove that $\text{sign}(f) = \text{sign}(a_k)$.

Let $g(x) = a_n x^{n-k} + \cdots + a_{k+1} x + a_k$. Then $f(x) = x^k g(x)$.

If $k = 0$, then $f(x) = g(x)$. Otherwise $f(x) \neq g(x)$, and since $\text{sign}(f) = \text{sign}(x^k) \text{sign}(g)$ and $\text{sign}(x^k) = 1$, it follows that $\text{sign}(f) = \text{sign}(g)$. We want $\text{sign}(g) = \text{sign}(a_k)$.

If $g(x) = a_k$ we are done. Otherwise let $h(x) = a_n x^{n-k-1} + \cdots + a_{k+2} x + a_{k+1}$. Then $g(x) = a_k + xh(x)$ and $h(x) \neq 0$. Since $|x^m| < 1$ for every $m \in \mathbb{N}$, we get

$$|h(x)| \leq |a_n| + \cdots + |a_{k+1}| := c > 0, \quad c \in \mathbb{R}.$$

Then

$$|xh(x)| \leq c|x| < |a_k|,$$

otherwise $|x| \geq \frac{|a_k|}{c}$, contradiction.

Therefore $\text{sign}(g) = \text{sign}(a_k + xh) = \text{sign}(a_k)$, as required (Note that one needs to verify that $|a| > |b| \Rightarrow \text{sign}(a + b) = \text{sign}(a)$).

□

We now want to classify all orderings on $\mathbb{R}(x)$ which make it into an ordered field. For this we need the notion of Dedekind cuts.

2. DEDEKIND CUTS

Notation 2.1. Let (Γ, \leq) be a totally ordered set and let $L, U \subseteq \Gamma$. If we write

$$L < U$$

we mean that

$$x < y \quad \forall x \in L, \forall y \in U.$$

(Similarly for $L \leq U$)

Definition 2.2. (*Dedekindschnitt*) Let (Γ, \leq) be a totally ordered set. A **Dedekind cut** of (Γ, \leq) is a pair (L, U) such that $L, U \subseteq \Gamma$, $L \cup U = \Gamma$ and $L < U$.

Remark 2.3. Since $L < U$ it follows that $L \cap U = \emptyset$. Therefore the subsets L, U form a partition of Γ (The letter "L" stands for "lower cut" and the letter "U" for "upper cut").

Example 2.4. Let (Γ, \leq) be a totally ordered set. For every $\gamma \in \Gamma$ we can consider the following two Dedekind cuts:

$$\begin{aligned} \gamma_- &:= (] - \infty, \gamma[, [\gamma, \infty[) \\ \gamma_+ &:= (] - \infty, \gamma],]\gamma, \infty[) \end{aligned}$$

Moreover if we take $L, U \in \{\emptyset, \Gamma\}$, then we have two more cuts:

$$-\infty := (\emptyset, \Gamma), \quad +\infty := (\Gamma, \emptyset)$$

Example 2.5. Consider the Dedekind cut (L, U) of (\mathbb{Q}, \leq) given by

$$L = \{x \in \mathbb{Q} : x < \sqrt{2}\} \quad \text{and} \quad U = \{x \in \mathbb{Q} : x > \sqrt{2}\}.$$

Then there is no $\gamma \in \mathbb{Q}$ such that $(L, U) = \gamma_-$ or $(L, U) = \gamma_+$.

Definition 2.6. (*trivialen und freie Schnitte*) Let (L, U) be a Dedekind cut of a totally ordered set (Γ, \leq) . If $(L, U) = \pm\infty$ or there is some $\gamma \in \Gamma$ such that $(L, U) = \gamma_+$ or $(L, U) = \gamma_-$ (as defined in 2.4), then (L, U) is said to be a **trivial** (or **realized**) Dedekind cut. Otherwise it is said to be a **free** Dedekind cut (or **gap**).

Remark 2.7. A Dedekind cut (L, U) of a totally ordered set (Γ, \leq) is free if $L \neq \emptyset, U \neq \emptyset, L$ has no last element and U has no least element.

Definition 2.8. (*Dedekindvollständigkeit*) A totally ordered set (Γ, \leq) is said to be **Dedekind complete** if for every pair (L, U) of subsets of Γ with $L \neq \emptyset, U \neq \emptyset$ and $L \leq U$, there exists $\gamma \in \Gamma$ such that

$$L \leq \gamma \leq U.$$

Exercise 2.9. Show that a totally ordered set (Γ, \leq) is Dedekind complete if and only if (Γ, \leq) has no free Dedekind cuts.

Examples 2.10.

- The ordered set of the reals (\mathbb{R}, \leq) is Dedekind complete, i.e. the set of Dedekind cuts of (\mathbb{R}, \leq) is $\{a_{\pm} : a \in \mathbb{R}\} \cup \{-\infty, +\infty\}$.
- We have already seen in 2.5 that (\mathbb{Q}, \leq) is not Dedekind complete. We can generalize 2.5: for every $\alpha \in \mathbb{R} - \mathbb{Q}$ we have the gap given by $(] - \infty, \alpha[\cap \mathbb{Q},]\alpha, \infty[\cap \mathbb{Q})$.

3. THE ORDERINGS ON $\mathbb{R}(x)$

Theorem 3.1. *There is a canonical bijection between the set of the orderings on $\mathbb{R}(x)$ and the set of the Dedekind cuts of \mathbb{R} .*

Proof. Let \leq be an ordering on $\mathbb{R}(x)$. Consider the sets $L = \{v \in \mathbb{R} : v < x\}$ and $U = \{w \in \mathbb{R} : x < w\}$. Then $\mathcal{C}_x^{\leq} := (L, U)$ is a Dedekind cut of \mathbb{R} . (Note that if \leq is the order defined in 1.1 then $\mathcal{C}_x^{\leq} = 0_+$). So we can define a map

$$\{\leq : \leq \text{ is an ordering on } \mathbb{R}(\mathbf{x})\} \xrightarrow{f} \{(L, U) : (L, U) \text{ is a Dedekind cut of } \mathbb{R}\}$$

$$\leq \quad \mapsto \quad \mathcal{C}_x^{\leq}$$

We now want to find a map
 $\{(L, U) : (L, U) \text{ is a Dedekind cut of } \mathbb{R}\} \longrightarrow \{\leq : \leq \text{ is an ordering on } \mathbb{R}(\mathbf{x})\}$

which is the inverse of f . Every Dedekind cut of (\mathbb{R}, \leq) is of the form $-\infty$, a_- , a_+ , $+\infty$, with $a \in \mathbb{R}$. With a change of variable, respectively, $y := -1/x$, $y := a - x$, $y := x - a$, $y := 1/x$, we obtain an ordering on $\mathbb{R}(y)$ such that

$$0 < y < r \quad \forall r \in \mathbb{R}, r > 0.$$

We have seen in 1.3 that there is only one ordering with such a property, so we have a well-defined map from the set of the Dedekind cuts of (\mathbb{R}, \leq) into the set of orderings of $\mathbb{R}(\mathbf{x})$. It is precisely the inverse of f . \square

4. ORDER PRESERVING EMBEDDINGS

Definition 4.1. (*ordnungstreue Einbettung*) Let (K, \leq) and (F, \leq) be ordered fields. An injective homomorphism of fields

$$\varphi: K \hookrightarrow F$$

is said to be an **order preserving embedding** if

$$a \leq b \Rightarrow \varphi(a) \leq \varphi(b) \quad \forall a, b \in K.$$

Theorem 4.2 (Hölder). *Let (K, \leq) be an Archimedean ordered field. Then there is an order preserving embedding*

$$\varphi: K \hookrightarrow \mathbb{R}.$$

Proof. Let $a \in K$. Consider the sets

$$I_a :=]-\infty, a]_K \cap \mathbb{Q} \quad \text{and} \quad F_a := [a, \infty[_K \cap \mathbb{Q}.$$

Then $I_a \leq F_a$ and $I_a \cup F_a = \mathbb{Q}$. So we can define

$$\varphi(a) := \sup I_a = \inf F_a \in \mathbb{R}.$$

Since K is Archimedean, φ is well-defined. Note that

$$I_a + I_b = \{x + y : x \in I_a, y \in I_b\} \subseteq I_{a+b}$$

and

$$F_a + F_b \subseteq F_{a+b},$$

then $\varphi(a) + \varphi(b) \leq \varphi(a + b)$ and $\varphi(a) + \varphi(b) \geq \varphi(a + b)$. This proves that φ is additive. Similarly one gets $\varphi(ab) = \varphi(a)\varphi(b)$. \square