

# The valuation difference rank of a quasi-ordered difference field

## §1 Quasi-ordered fields $\leftarrow$ joint setting for both ordered & valued fields!

Def:  $(S, \leq)$  is a q.o. set if  $\leq$  is a reflexive and transitive total operation.

Equiv.: for all  $a, b \in S$ ;  $a \approx b$  iff  $a \leq b$  and  $b \leq a$ .

$\Delta$  q.o. field  $(K, \leq)$  is a q.o. set i.h.

①  $a \approx a \Rightarrow a = 0$

②  $0 \leq c$  and  $a \leq b \Rightarrow ac \leq bc$

③ if  $a \leq b$  and  $b \neq c$  then  $a+c \leq b+c$ .

Examples: ②  $(K, \leq)$  totally ordered fields

③  $(K, v)$  valued field:  $a \leq b$  iff  $v(a) \leq v(b)$

Thm of Takhraddin: any q.o. field is either ② or ③ proper quasi-order  $\rightarrow$

$E_1 :=$  equiv. class of  $1 \in K$  wrt  $\approx$

If  $E_1 = \{1\}$ , then ② if  $E_1 \neq \{1\} \Rightarrow E_1 = \mathcal{O}_v^\times$ .

Convex valuations: Let  $(K, \leq)$  be q.o.,  $w$  a valuation is convex if  $\mathcal{O}_w$  is convex

Compatible valuations:  $w$  is compatible wrt  $\leq$  if  $0 \leq a \leq b \Rightarrow w(a) \leq w(b)$

Theorem:  $w$  is compatible iff  $\mathcal{O}_w$  is convex  
iff  $m_w$  is convex iff  $m_w < 1$  iff  
 $\leq$  naturally induces via the residue map a q.o. on  $K_w$ .

Let  $(K, \leq)$  be a q.o. field,  $\leq$  the natural valuation, i.e.  $v$  is the finest  $\leq$ -convex valuation.

$\mathcal{R} :=$  order type of strict coarsenings of the natural valuation

## §2: Descent

$G := v(K^\times)$ . On  $G^{<0}$  endowed with the arch. equiv. relation,  $G^{<0} / \sim_{\text{arch}} =: \Gamma$ ,  $\Gamma$  is the rank of  $G$ .

Define  $V_G: G^{<0} \rightarrow \Gamma$   
 $g \mapsto [g]_{\text{arch}}$

Let  $w$  be a valuation on  $(K, \leq)$ , define  $G_w := v(G_w^\times)$  a convex subgroup of  $G$ .

Lemma 1:  $G_w \rightarrow G_w$  is an order-preserving bijection from  $\mathcal{R} \rightarrow \text{rank } G$

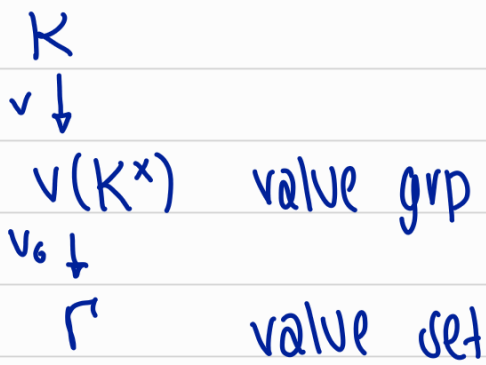
Lemma 2:  $G_w \rightarrow \Gamma_w$  is a bij. correspondence from  $\text{rank}(G) \rightarrow \underbrace{\Gamma_{fs}}_{\text{l.o. set of non-empty finite segments}}$

Theorem:  $\mathcal{R} \xrightarrow{\sim} \Gamma_{fs}$

principal finite segment:  $[g, +\infty)$   
 $\Gamma_{fs} \cong \Gamma^*$  where  $\Gamma^*$  is the reversely l.o. set.

Def: Set  $P_K := K^{\geq 0} \setminus \{0\}$ .  $G_w$  is principal if ex.  $a \in P_K$  s.t.h.  $G_w$  is the smallest convex subring containing  $a$ .

## Descent



**Theorem:**  $\mathbb{R}^{pr} \cong \Gamma^*$ .

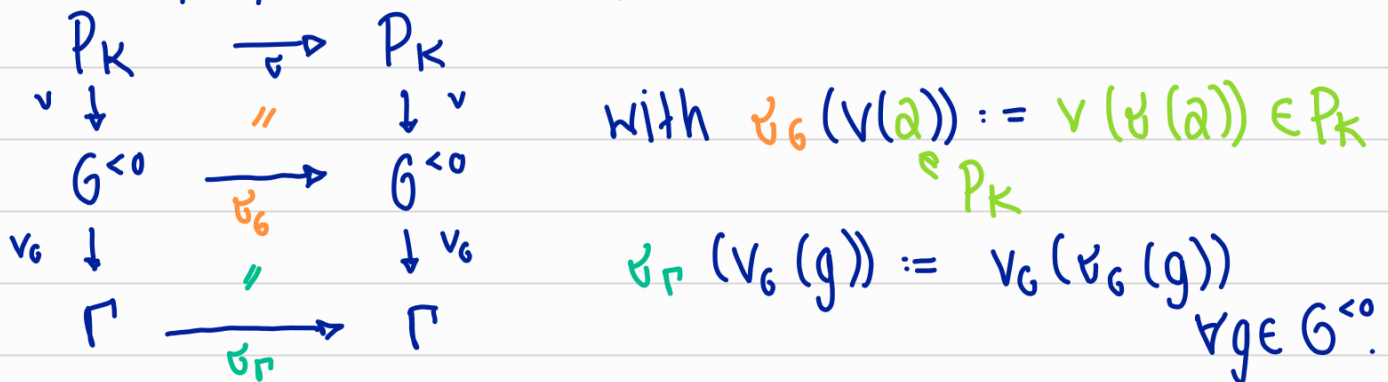
**Example:** Let  $\tau$  be an order type. Then there is a maximal valued Hahn field of p.r.  $\tau$ .  
 [construction:  $\mu := \tau^*$   
 $G = \text{Hahn group with value set } \tau^*$ ]

## Part 3: Quasi-ordered difference fields

$(K, \preceq, \sigma)$  q.o. difference field,  $\sigma \in \text{Aut}(K)$

**Def:** Let  $(K, \preceq)$  be a q.o. field.  $\sigma \in \text{Aut}(K)$  is said to be **q.o. preserving** if  
 $a \preceq a' \iff \sigma(a) \preceq \sigma(a')$

Given  $(K, \preceq, \sigma)$ . What is the descent behaviour?



Let  $w$  be a convex valuation. We say  $w$  is  **$\sigma$ -compatible** iff  
 $w(a) \leq w(b) \iff w(\sigma(a)) \leq w(\sigma(b))$

**Problem:** characterize  $\mathcal{R}_\sigma := \text{ord. type of } \sigma\text{-comp. val.}$

**Theorem:**  $W$  is  $\sigma$ -compatible  $\Leftrightarrow \sigma(G_W) = \mathcal{O}_W$

$$\Leftrightarrow \sigma(m_W) = m_W$$

$\Leftrightarrow$  the map  $\sigma_W: K_W \rightarrow K_W, a_W \rightarrow \sigma(a)_W$  is a well-defined q.o. preserving autom of the quasi-ordered field  $(K_W, \leq)$ .

$\sigma_G$ -rank: order type of  $\sigma_G$ -invariant convex subgrps of  $G$

$\sigma_r - \Gamma^{fs}$ : order type of  $\sigma_r$ -inv. final segments

**Lemma 1:**  $\mathcal{O}_W \xrightarrow{\text{bij.}} G_W$  is an order-preserving bij. from  $\mathcal{R}_\sigma$  onto the  $\sigma_G$ -rank of  $G$ .

**Lemma 2:**  $G_W \xrightarrow{\text{bij.}} \Gamma_W$  from the  $\sigma_G$ -rank of  $G$  onto  $\sigma_r - \Gamma^{fs}$ .

**Theorem:**  $\mathcal{O}_W \xrightarrow{\text{bij.}} \Gamma_W$  is an ord.-pres. bij. from  $\mathcal{R}_\sigma \xrightarrow{\text{bij.}} \sigma_r - \Gamma^{fs}$ .

! From now on, assume  $\sigma(a) > a$ . !

In this case, we have 3 convex equivalence rel.

$$a \sim_\sigma a' \quad \text{iff} \quad v(a) \sim_{\sigma_G} v(a') \quad \text{iff} \\ v_G(v(a)) \sim_{\sigma_r} v_G(v(a'))$$

**Corollary:**  $\mathcal{R}_\sigma^{\text{p.r.}}$  is isomorphic to  $(\Gamma / \sim_{\sigma_r})^*$

Corollary (Hahn construction): Given an order type  $\tau$ , there is a maximal q.o. field and  $\mathcal{O}$   $\leq$ -comp.

s.t.h.  $\mathcal{R}_{\mathcal{O}}^{\text{p.r.}} = \tau$ .

Pf: Set  $\mu = \tau^*$ ,  $\Gamma := \sum \mathbb{Q}$   
Let  $\ell$  be the autom.  $^{\mu} q \mapsto q+1$  on every copy of  $\mathbb{Q}$ .

Let  $\mathcal{O}_{\Gamma} := \sum_{\gamma \in \mu} \ell_{\gamma}$ ,  $G = \bigoplus_{\Gamma} \mathbb{R} \sim \mathcal{O}_G$

$\mathbb{K} := \mathbb{R}(\langle G \rangle) \sim \mathcal{O}$ . □

Question (Simone): Also for non-surjective  $\mathcal{O}$ ?

Question (Elliot): Can you do both at the same time?

Question (Salma):  $\mathcal{O}$ -rank is an invariant.

What happens if we fix  $\mathcal{O}$ , and vary  $v$ ?

$\leadsto$  get colouring of autom's

Is the resulting map interesting at all?