The valuation difference rank of a quasi-ordered difference field

§1 Quasi-ordered fields ← joint setting for both ordered & valued fields!

Def: $(S, \precsim)$ is a q.o. set if $\precsim$ is a reflexive and transitive total operation.

Equiv.: for all $a, b \in S$; $a \approx b$ iff $a \precsim b$ and $b \precsim a$.

A q.o. field $(K, \precsim)$ is a q.o. set s.th.

1. $a \approx a \Rightarrow a = 0$
2. $0 \precsim c$ and $a \precsim b \Rightarrow ac \precsim bc$
3. if $a \precsim b$ and $b \not\approx c$ then $a + c \precsim b + c$.

Examples: (a) $(K, \leq)$ totally ordered fields

(b) (K, v) valued field: $a \precsim b$ iff $v(a) \leq v(b)$

Thm of Fakhruddin: any q.o. field is either (a) or (b)

proper quasi-order

$E_1 :=$ equiv. class of $1 \in K$ wrt $\approx$

If $E_1 = \{1\}$, then (a) if $E_1 \neq \{1\} \Rightarrow E_1 = \mathcal{O}_v^\times$.

Convex valuations: Let $(K, \precsim)$ be q.o., w a valuation is convex if $\mathcal{O}_w$ is convex

Compatible valuations: w is compatible wrt $\precsim$ if $0 \precsim a \precsim b \Rightarrow w(a) \leq w(b)$

Theorem: w is compatible iff $\mathcal{O}_w$ is convex iff $\mathcal{M}_w$ is convex iff $\mathcal{M}_w < 1$ iff $\precsim$ naturally induces via the residue map a q.o. on Kw.

Let $(K, \leq)$ be a q.o. field, $\leq$ the natural valuation, i.e. v is the finest $\leq$-convex valuation.

$\mathcal{R} :=$ order type of strict coarsenings of the natural valuation

§2: Descent

$G := v(K^\times)$. On $G^{<0}$ endowed with the arch. equiv. relation, $G^{<0}/\sim_{arch} =: \Gamma$, Γ is the rank of G.

Define $v_G : G^{<0} \twoheadrightarrow \Gamma$

$$g \longmapsto [g]_{arch}$$

Let w be a valuation on $(K, \leq)$, define $G_w := v(\mathcal{O}_w^\times)$ a convex subgroup of G.

Lemma 1: $\mathcal{O}_w \to G_w$ is an order-preserving bijection from $\mathcal{R} \to \operatorname{rank} G$

Lemma 2: $G_w \to \Gamma_w$ is a bij. correspondence from $\operatorname{rank}(G) \to \underbrace{\Gamma_{fs}}_{}$

l.o. set of non-empty finite segments

Theorem: $\mathcal{R} \xrightarrow{\sim} \Gamma_{fs}$

principal finite segment: $[\gamma, +\infty)$

$\Gamma^{pfs} \simeq \Gamma^*$ where Γ^* is the reversely l.o. set.

Def: Set $P_K := K^{>0} \setminus \mathcal{O}_v$. $\mathcal{O}_w$ is principal if ex. $a \in P_K$ s.th. $\mathcal{O}_w$ is the smallest convex subring containing a.

Descent

$$\begin{tikzcd} K \arrow[d, "v"'] & \\ v(K^\times) \arrow[d, "v_G"'] & \text{value grp} \\ \Gamma & \text{value set} \end{tikzcd}$$

Theorem: $\mathbb{R}^{pr} \simeq \Gamma^*$.

Example: Let τ be an order type. Then there is a maximal valued Hahn field of p.r. τ.
[construction: $\mu := \tau^*$
G = Hahn group with value set τ^*]

Part 3: Quasi-ordered difference fields

$(K, \precsim, \sigma)$ q.o. difference field, $\sigma \in \mathrm{Aut}(K)$

Def: Let $(K, \precsim)$ be a q.o. field. $\sigma \in \mathrm{Aut}(K)$ is said to be q.o. preserving if
$$a \precsim a' \iff \sigma(a) \precsim \sigma(a')$$

Given $(K, \precsim, \sigma)$. What is the descent behaviour?

$$\begin{tikzcd} P_K \arrow[r, "\sigma"] \arrow[d, "v"'] \arrow[dr, phantom, "/\!/"] & P_K \arrow[d, "v"] \\ G^{<0} \arrow[r, "\sigma_G"'] \arrow[d, "v_G"'] \arrow[dr, phantom, "/\!/"] & G^{<0} \arrow[d, "v_G"] \\ \Gamma \arrow[r, "\sigma_\Gamma"'] & \Gamma \end{tikzcd}$$

with $\sigma_G(v(a)) := v(\sigma(a)) \in P_K$
$\in P_K$

$\sigma_\Gamma(v_G(g)) := v_G(\sigma_G(g))$
$\forall g \in G^{<0}$.

Let w be a convex valuation. We say w is σ-compatible iff
$$w(a) \leq w(b) \iff w(\sigma(a)) \leq w(\sigma(b))$$

Problem: characterize $\mathcal{R}_\sigma :=$ ord. type of σ-comp. val.

Theorem: w is σ-compatible $\Leftrightarrow$ $\sigma(\mathcal{O}_w) = \mathcal{O}_w$
$\Leftrightarrow$ $\sigma(\mathfrak{m}_w) = \mathfrak{m}_w$
$\Leftrightarrow$ the map $\sigma_w : Kw \to Kw$, $aw \mapsto \sigma(a)w$
is a well-defined q.o. preserving autom of the quasi-ordered field $(Kw, \preccurlyeq)$.

σ_G-rank: order type of σ_G-invariant convex subgrps of G

$\sigma_\Gamma - \Gamma^{fs}$: order type of σ_Γ-inv. final segments

Lemma 1: $\mathcal{O}_w \mapsto G_w$ is an order-preserving bij. from $\mathcal{R}_\sigma$ onto the σ_G-rank of G.

Lemma 2: $G_w \mapsto \Gamma_w$ —"— from the σ_G-rank of G onto $\sigma_\Gamma - \Gamma^{f.s.}$

Theorem: $\mathcal{O}_w \to \Gamma_w$ is an ord.-pres. bij. from $\mathcal{R}_\sigma \to \sigma_\Gamma - \Gamma^{f.s.}$

⚠ From now on, assume $\sigma(a) > a$. ⚠

In this case, we have 3 convex equivalence rel.
$a \sim_\sigma a'$ iff $v(a) \sim_{\sigma_G} v(a')$ iff
$v_G(v(a)) \sim_{\sigma_\Gamma} v_G(v(a'))$

Corollary: $\mathcal{R}_\sigma^{p.r.}$ is isomorphic to $(\Gamma/\sim_{\sigma_\Gamma})^*$

Corollary (Hahn construction): Given an order type τ, there is a maximal q.o. field and σ $\precsim$-comp. s.th. $R_\sigma^{p.r.} = \tau$.

Pf: Set $\mu = \tau^*$, $\Gamma := \sum_\mu \mathbb{Q}$

Let ℓ be the autom. $q \mapsto q+1$ on every copy of $\mathbb{Q}$.

Let $\sigma_\Gamma := \sum_{\gamma \in \mu} \ell_\gamma$, $\quad G = \bigoplus_\Gamma \mathbb{R} \rightsquigarrow \sigma_G$

$\mathbb{K} := \mathbb{R}((G)) \rightsquigarrow \sigma$. $\quad \boxtimes$

Question (Simone): Also for nonsurjective σ?

Question (Elliot): Can you do both at the same time?

Question (Salma): σ-rank is an invariant. What happens if we fix σ, and vary v?
$\rightsquigarrow$ get colouring of autom.'s
Is the resulting map interesting at all?