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Contribution by Salma Kuhlmann, Universität Konstanz, Germany

M. Ghasemi, S. Kuhlmann, M. Marshall; *Moment problem in infinitely many variables*, Israel J. Math., **212**, 989-1012 (2016)

Moment problem in infinitely many variables

SUMMARY

In this talk we present the moment problem for the polynomial algebra $A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$ in an arbitrary number of variables x_i, $i \in \Omega$. We introduce constructibly Radon measures on its character space $\mathbb{R}^\Omega$, and proceed to investigate their relationship to positive linear functionals on A. The main tools are algebraic; we exploit the localisation techniques introduced by Marshall, and work with the localisation $B = B_\Omega := \mathbb{R}[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega]$ of A. We show that positive linear functionals on B_Ω correspond bijectively to constructibly Radon measures on $\mathbb{R}^\Omega$, and that the moment problem for A_Ω reduces to understanding the extensions of a positive linear functional on A_Ω to a positive linear functional on B_Ω.

We observe that A (resp. B) is the inductive limit of the $\mathbb{R}$-algebras A_I (resp., B_I), I running through all finite subsets of Ω. So many questions about A and B reduce to the case where Ω is finite. This last observation is exploited and formalised as the *projective limit approach* to the moment problem (see M. Infusino 's talk), connecting in particular constructibly Radon measures to cylindrical measures.

M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski; *Projective limits techniques for the infinite dimensional moment problem*, submitted (2020)

History: The univariate moment problem

Is an old problem with origins tracing back to work of Stieltjes. Given a sequence $(s_k)_{k\geq 0}$ of real numbers one wants to know when there exists a (positive) Radon measure μ on $\mathbb{R}$ such that

$$s_k = \int x^k d\mu \,\forall\, k \geq 0.$$

Since the monomials $x^k, k \geq 0$ form a basis for the polynomial algebra $\mathbb{R}[x]$, this problem is equivalent to the following one: Given a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, when does there exist a Radon measure μ on $\mathbb{R}$ such that $L(f) = \int f d\mu \,\forall\, f \in \mathbb{R}[x]$. Akhiezer 1965 and Shohat-Tamarkin 1943 are standard references.

History: The multivariate moment problem

Has been considered more recently. For $n \geq 1$, $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables $x_1, \ldots, x_n$. Given a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ and a closed subset Y of $\mathbb{R}^n$ one wants to know when there exists a Radon measure μ on $\mathbb{R}^n$ supported on Y such that $L(f) = \int f d\mu \ \forall$ $f \in \mathbb{R}[\underline{x}]$.

Haviland, 1936

Such a measure exists if and only if $L(\mathrm{Pos}(Y)) \subseteq [0, \infty)$, where $\mathrm{Pos}(Y) := \{f \in \mathbb{R}[\underline{x}] : f(x) \geq 0 \quad \forall x \in Y\}$.

Berg 1987, Fuglede 1983 are general references. A major motivation here is the close connection between the multivariate moment problem and real algebraic geometry; Schmüdgen 1999, Putinar 2000, Marshall 2008, Lasserre 2013.

I. WHAT is the infinite-variate moment problem?

- We consider a commutative unital $\mathbb{R}$-algebra A,
- its character space $X(A)$, which is the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$ (sending 1 to 1).
- The only ring homomorphism from $\mathbb{R}$ to itself is the identity.
- For $a \in A$, $\hat{a} = \hat{a}_A : X(A) \to \mathbb{R}$ is defined by $\hat{a}_A(\alpha) = \alpha(a)$.
- $X(A)$ is given the weakest topology such that the functions $\hat{a}_A$, $a \in A$ are continuous.
- For a topological space X, $C(X)$ denotes the ring of all continuous functions from X to $\mathbb{R}$.
- The mapping $a \mapsto \hat{a}_A$ defines a ring homomorphism from A into $C(X(A))$.

EXAMPLES

- Ring homomorphisms from the polynomial ring in n variables $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^n$, and
- $X(\mathbb{R}[\underline{x}])$ is identified as a topological space with $\mathbb{R}^n$.
- Let Ω is an arbitrary index set. Ring homomorphisms from the polynomial ring $A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$ to $\mathbb{R}$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$, and
- $X(A_\Omega) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.
- Two further important examples will be considered in the next sections.

- $\sum A^2$ denotes the cone of all finite sums $\sum a_i^2$, $a_i \in A$.
- For a subset X of $X(A)$, we define

$$\mathrm{Pos}_A(X) := \{a \in A \mid \hat{a}_A \geq 0 \text{ on } X\}.$$

- A linear functional $L : A \to \mathbb{R}$ is said to be positive if $L(\sum A^2) \subseteq [0, \infty)$ and strongly-positive if $L(\mathrm{Pos}_A(X(A))) \subseteq [0, \infty)$.

The general moment problem I

- A Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of Borel sets of $X(A)$ which is locally finite (every point has a neighbourhood of finite measure) and inner regular (each Borel set can be approximated from within using a compact set).
- For a linear functional $L : A \to \mathbb{R}$, we consider the set of Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu \ \forall\ a \in A$.
- If such a measure exists for L, we call it a representing measure.
- When does L admit a representing Radon measure?

The following version of Haviland's Theorem (to get representing Radon measures) was proven in [Marshall, *Approximating positive polynomials using sums of squares*, Can. Math. Bull. 46, 400-418 (2003)]:

Theorem [Marshall]

Suppose A is an $\mathbb{R}$-algebra, X is a Hausdorff space, and $\hat{} : A \longrightarrow C(X)$ is an $\mathbb{R}$-algebra homomorphism such that for some $p \in A$, $\hat{p} \geq 0$ on X, the set $X_i = \hat{p}^{-1}([0, i])$ is compact for each $i = 1, 2, \cdots$. Then for every linear functional $L : A \longrightarrow \mathbb{R}$ satisfying $L(\mathrm{Pos}_A(X)) \subseteq [0, \infty)$, there exists a Radon measure μ on X such that $\forall a \in A \quad L(a) = \int_X \hat{a}\, d\mu$.

Remarks

- The theorem applies to $A = \mathbb{R}[\underline{x}]$ and $X = \mathbb{R}^n$,with $p := \sum x_i^2$ and $\hat{p}(x) = ||x||^2$. This gives the classical Haviland result: a strongly positive linear functional on $\mathbb{R}[\underline{x}]$ admits a representing Radon measure.
- The hypothesis of the theorem implies in particular that X is locally compact.
- In particular, the theorem does not apply to $A_\Omega = \mathbb{R}[x_i] \mid i \in \Omega]$ and $X = \mathbb{R}^\Omega$, if Ω is infinite.

Constructibly Borel subsets of $X(A)$

- Let A be an $\mathbb{R}$-algebra. The open sets
$$U_A(a) := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) > 0\},\ a \in A$$
form a basis for the topology on $X(A)$
- If A is generated as an $\mathbb{R}$-algebra by x_i, $i \in \Omega$, the embedding $X(A) \hookrightarrow \mathbb{R}^\Omega$ defined by $\alpha \mapsto (\alpha(x_i))_{i\in\Omega}$ identifies $X(A)$ with a subspace of $\mathbb{R}^\Omega$.
- Sets of the form
$$\{b \in \mathbb{R}^\Omega \mid \sum_{i\in I}(b_i - p_i)^2 < r\},$$
where $r, p_i \in \mathbb{Q}$ and I is a finite subset of Ω, form a basis for the product topology on $\mathbb{R}^\Omega$.

- It follows that sets of the form

$$U_A(r - \sum_{i \in I} (x_i - p_i)^2),\ r, p_i \in \mathbb{Q},\ I \text{ a finite subset of } \Omega, \quad (1)$$

  form a basis for the topology on $X(A)$.
- A subset E of $X(A)$ is called Borel if E is an element of the σ-algebra of subsets of $X(A)$ generated by the open sets.
- A subset E of $X(A)$ is said to be constructible or semialgebraic (resp., constructibly Borel) if E is an element of the algebra (resp., σ-algebra) of subsets of $X(A)$ generated by $U_A(a)$, $a \in A$.
- Constructible $\Rightarrow$ constructibly Borel $\Rightarrow$ Borel.

Two important observations:

Countably generated algebras

If A is generated as an $\mathbb{R}$-algebra by a countable set $\{x_i \mid i \in \Omega\}$ then every Borel set of $X(A)$ is constructibly Borel.

- Proof: Sets as in (1) form a countable basis for the topology on $X(A)$.

Pull backs to countably generated subalgebras

A subset E of $X(A)$ is constructibly Borel iff $E = \pi^{-1}(E')$ for some Borel set E' of $X(A')$, where A' is a countably generated subalgebra and $\pi : X(A) \to X(A')$ is the restriction map.

- Proof: Clearly $U_A(a) = \pi^{-1}(U_{A'}(a))$ for any $a \in A'$. This implies that, for each Borel set E' of $X(A')$, $\pi^{-1}(E')$ is an element of the σ-algebra $\Sigma_{A'}$ (consisting of subsets of $X(A)$ generated by the $U_A(a)$, $a \in A'$). Now observe that $\cup\Sigma_{A'}$ (A' running through the countably generated subalgebras of A) is itself a σ-algebra.

Constructibly Radon measures

A constructibly Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of constructibly Borel sets of $X(A)$ such that for, each countably generated subalgebra A' of A, the pushforward ν of μ to $X(A')$ via the restriction map $\pi : \alpha \mapsto \alpha|_{A'}$ is a Radon measure on $X(A')$.For a Borel Z of $X(A')$ define $\nu(Z) := \mu(\pi^{-1}(Z))$.

From now on we consider only Radon and constructibly Radon measures having the additional property that $\hat{a}_A$ is μ-integrable (i.e., $\int \hat{a}_A d\mu$ is well-defined and finite) for all $a \in A$.

THE GENERAL MOMENT PROBLEM II

For a linear functional $L : A \to \mathbb{R}$, we consider the set of Radon or constructibly Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu \; \forall \, a \in A$. The moment problem is to understand this set of measures. We are particularly interested in representing positive or strongly positive linear functionals. In the following we shall solve the problem for...

Three special algebras

Let Ω is an arbitrary index set.

- As above, $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, we further define
- $B = B_\Omega := \mathbb{R}[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega]$, the localization of A at the multiplicative set generated by the $1 + x_i^2$, $i \in \Omega$, and
- $C = C_\Omega := \mathbb{R}[\frac{x_i}{1+x_i^2}, \frac{1}{1+x_i^2} \mid i \in \Omega]$, the $\mathbb{R}$-subalgebra of B generated by the elements $\frac{1}{1+x_i^2}$, $\frac{x_i}{1+x_i^2}$, $i \in \Omega$.

III. HOW?

- By definition, A (resp., B, resp., C) is the direct limit of the $\mathbb{R}$-algebras A_I (resp., B_I, resp., C_I), I running through all finite subsets of Ω.
- These algebras were studied extensively in [Marshall 2003] for finite Ω. Because of this, the results in the next 4 slides regarding A, B and C use Marshall's corresponding results for the case where Ω is finite.

Character Spaces

- C is naturally identified (via $y_i \leftrightarrow \frac{1}{1+x_i^2}$ and $z_i \leftrightarrow \frac{x_i}{1+x_i^2}$) with the polynomial algebra $\mathbb{R}[y_i, z_i \mid i \in \Omega]$ factored by the ideal generated by the polynomials $y_i^2 + z_i^2 - y_i = (y_i - \frac{1}{2})^2 + z_i^2 - \frac{1}{4}, i \in \Omega$.
- Consequently, $X(C)$ is compact, indeed it is identified naturally with $\mathbb{S}^\Omega$, where $\mathbb{S} := \{(y, z) \in \mathbb{R}^2 \mid (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}\}$.
- $\sum C^2$ is Archimedean.
- The restriction map $\alpha \mapsto \alpha|_C$ identifies $X(B)$ with a subspace of $X(C)$. In terms of coordinates, this map is given by $\alpha = (x_i)_{i\in\Omega} \mapsto \beta = (y_i, z_i)_{i\in\Omega}$, where $y_i := \frac{1}{1+x_i^2}$, $z_i := \frac{x_i}{1+x_i^2}$. In particular, the image of $X(B)$ is dense in $X(C)$.

- Elements of $X(A)$ and $X(B)$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$.
- $X(A) = X(B) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.
- $X(C) \setminus X(B) = \cup_{i \in \Omega} \Delta_i$ where $\Delta_i := \{\beta \in X(C) \mid \beta(\frac{1}{1+x_i^2}) = 0\}$.

Positivity

- A linear functional $L : C \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_C(X(C))$-positive (i.e. strongly positive).
- a linear functional $L : B \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_B(X(B))$-positive (i.e. strongly positive).
- For linear functionals $L : A \to \mathbb{R}$ this is never the case (except for the univariate case), but we have:

Extendibility from A to B

For a linear functional $L : A \to \mathbb{R}$, L is an $\mathrm{Pos}_A(X(A))$-positive (i.e. strongly positive) iff L extends to a positive linear functional $L : B \to \mathbb{R}$.

IV: RESULTS

Moment Problem for C

Positive linear functionals $L : C \to \mathbb{R}$ are in natural one-to-one correspondence with Radon measures μ on the compact space $X(C)$ via $L \leftrightarrow \mu$ iff $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$.

Main Lemma

For each positive linear functional $L : B \to \mathbb{R}$ there exists a unique Radon measure μ on $X(C)$ such that $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$. This satisfies $\mu(\Delta_i) = 0 \; \forall \, i \in \Omega$ and $L(f) = \int \tilde{\hat{f}} d\mu \; \forall f \in B$.

Moment Problem for B

There is a canonical one-to-one correspondence $L \leftrightarrow \nu$ given by $L(f) = \int \hat{f}_B d\nu \; \forall f \in B$ between positive linear functionals L on B and constructibly Radon measures ν on $X(B)$.

Main Corollary: Moment Problem for A

For any linear functional $L : A_\Omega \to \mathbb{R}$, the set of constructibly Radon measures ν on $\mathbb{R}^\Omega$ satisfying $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$ is in natural one-to-one correspondence with the set of positive linear functionals $L' : B_\Omega \to \mathbb{R}$ extending L.

Extension of Haviland's theorem

Let $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, the ring of polynomials in an arbitrary number of variables x_i, $i \in \Omega$ with coefficients in $\mathbb{R}$.

Extension of Haviland

Suppose $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$ satisfying condition (i) below. Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$.

Condition (i): Y is described by countably many inequalities i.e., there exists a countable $S \subset A_\Omega$ such that $Y = \{\alpha \in \mathbb{R}^\Omega \mid \hat{g}(\alpha) \geq 0 \; \forall g \in S\}$. We note that Condition (i) is always satisfied for countable Ω.

Extension of Haviland in the countable case

Suppose Ω is countable, $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$. Then there exists a Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$.

The cylinder σ-algebra and cylindrical measures

In [Infusino - Kuhlmann - Kuna - Michalski, 2020] we represent the character space as projective limit and indeed prove that our constructibly Radon measures are just cylindrical measures introduced in [L. Schwartz, Radon measures on arbitrary topological spaces and cylindrical measures. Tata Institute of Fundamental Research Studies in Mathematics, No. 6. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973.]