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http://math.usask.ca/~skuhlman/slidesiwota2010.pdf

Closures of Quadratic Modules in Locally Convex Topologies.

Positivity and Duality.

• Let $V := \mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the real vector space of polynomials in n variables and real coefficients.

• Fix τ a locally convex topological vector space topology on V. Denote V_τ the corresponding topological space.

• Let $K \subset \mathbb{R}^n$. Consider polynomials positive semi-definite on K:

$$\mathrm{Pos}(K) := \{f \in V \mid f(x) \geq 0 \text{ for all } x \in K\}$$

• Let $C \subset V$. Define

$$K_C := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \ \forall \ g \in C\}$$

• Let $C \subset V$. Define the dual of C:

$$C^{\vee} := \{L \mid L : V_{\tau} \to \mathbb{R}\,; \text{cts linear functional}; L(C) \geq 0\}$$

and the double dual of C:

$$C^{\vee\vee} := \{f \in V \mid L(f) \geq 0 \;\; \forall \;\; L \in C^{\vee}\}$$

Straightforward properties are:

(i) Contravariance

(ii) $C \subset C^{\vee\vee}$
(iii) $C^{\vee\vee\vee} = C^{\vee}$

etc...

The Moment Property and the Strong Moment Property.

Haviland's representation theorem for multidimensional moment sequences:

Let $K \subset \mathbb{R}^n$ closed, and $L : V \to \mathbb{R}$ a linear functional $\neq 0$. The following are equivalent:

(i) $L(f) \geq 0$ for all $f \in \mathrm{Pos}(K)$

(ii) $\exists$ a positive Borel measure μ on K such that

$$L(f) = \int_K f d\mu \, , \forall \, f \in V$$

We use Haviland's theorem and the properties of duality to deduce the following:

Theorem 1: Let V_τ as above, $C \subset V$, $K \subset \mathbb{R}^n$, K closed. The following are equivalent:

(1) $C^\vee \subset \mathrm{Pos}(K)^\vee$

(2) $C^{\vee\vee} \supset \mathrm{Pos}(K)$

(3) $\forall\, L \in C^\vee\ \exists\ \mu$ on K such that:

$$L(f) = \int_K f d\mu\ , \forall\, f \in V$$

Definitions: C satisfies (or solves) K MP if any of the equivalent conditions of the above theorem hold. C satisfies (or solves) the strong K MP if C satifies K MP with $K = K_C$.

Remark: These notions were introduced and studied by Schmüdgen for $\tau = \varphi :=$ the finest locally convex topology, thus studying representation of arbitrary linear functionals. Here we consider representation of τ continuous linear functionals. We shall return to this issue later.

Double duals and closures.

If $C \subset V$ is a (convex) cone (closed under addition and scalar multiplication by positive reals), then

$$C^{\vee\vee} = \overline{C}$$

in V_τ (Hahn–Banach). We obtain the following:

Corollary 1 Let $C \subset V$ be a cone, $K \subset \mathbb{R}^n$ closed. The following are equivalent:

(2) $\overline{C} \supset \mathrm{Pos}(K)$

(3) $\forall\, L \in C^\vee \;\exists\; \mu$ on K such that:

$$L(f) = \int_K f d\mu \,, \forall\, f \in V$$

These results are particularly interesting in the special case when C is a finitely generated quadratic module as we shall explain now.

Finite solvability of the K Moment Problem for continuous linear functionals.

Let $S := \{g_1, \cdots, g_s\} \subset V$. We define the (finitely generated) quadratic module:

$$M_S := \left\{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_i \in \sum V^2\right\} .$$

For S, M_S, K as above we obtain the following:

Corollary 2: The following are equivalent:

(2) $\overline{M_S} \supset \operatorname{Pos}(K)$

(3) If L is a continuous linear functional s.t.

$$L(h^2) \geq 0, L(h^2 g_1) \geq 0, \cdots, L(h^2 g_s) \geq 0$$

(for all $h \in V$), then there $\exists\ \mu$ on K such that:

$$L(f) = \int_K f d\mu \,, \forall\, f \in V .$$

Thus existence of representation via measures amounts to checking psd-ness of finitely many Hankel matrices.

Definition: The K MP is finitely solvable if S finite exists such that any of the quivalent conditions of Corollary 2 holds.

Remark: If $n \geq 2$ and K contains a 2-dimensional affine cone, the K MP is never finitely solvable for the finest topology φ (K–Marshall). The hope with this approach is to get finite solvability for representation of linear functionals continuous in a coarser topology τ. We discuss this now.

Closures of the cone of Sums of Squares

Theorem 2:

(1)

$$\overline{\sum V^2} = \sum V^2 \text{ in } V_\varphi \,.$$

(2)

$$\overline{\sum V^2} = \text{ Pos } [-1,1]^n \text{ in } V_p \,.$$

Here, for $1 \leq p \leq \infty$:

$V_P \; := \; V$ endowed with the ℓ_p–norm topology (on the coefficients of polynomials).

Two aspects: **a.** *Approximating nonnegative polynomials on the closed hypercube by sums of squares, and* **b.** *applications to solvability of MP:*

Corollary 2 and this theorem, say for $p = 1$, establish the following:

Corollary 3: Let L be a continuous linear functional on V_1, i.e. L is a linear functional on V with a bounded sequence of moments $(L(x^\alpha))_{\alpha\in\mathbb{N}^n}$.

Assume that

$$L(h^2) \geq 0 \ \forall \ h \in V .$$

Then

$$\exists \mu \text{ on } [-1,1]^n \text{ such that } L(f) = \int f d\mu \ \forall \ f \in V .$$

Closures in Weighted ℓ_p Topologies.

Let $1 \le p < \infty$, $r = (r_1, \cdots, r_n)$ be a n-tuple of positive real numbers.

• Set

$$\ell_p(r, \mathbb{N}^n) = \{s \in \mathbb{R}^{\mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r^\alpha < \infty\}$$

endowed with the norm defined by

$$\|(s)\|_{p,r} = (\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r^\alpha)^{\frac{1}{p}}.$$

• Denote by $V_{p\,,\,r}$ the topological vector space V endowed with the $\|\cdot\|_{p,r}$ norm.

• We compute the closure of the cone of sums of squares in these norm topologies.

Let $f \in V$. Assume that

$$f \geq 0 \text{ on } \prod_{i=1}^{n} [-r_i, r_i] \, .$$

Then the polynomial $\tilde{f}(\underline{X}) = f(r_1 X_1, \cdots, r_n X_n)$ is a non-negative polynomial on $[-1, 1]^n$.

Combining this observation with Berg's result we get:

Theorem 3

$$\overline{\sum V^2} \;=\; \text{Pos}\,(\prod_{i=1}^{n} [-r_i, r_i])$$

in $V_{1\,,\,r}$.

Theorem 4 For $1 < p < \infty$,

$$\overline{\sum V^2} = \text{Pos}\,(\prod_{i=1}^{n} [-r_i^{\frac{q}{p}}, r_i^{\frac{q}{p}}])$$

in $V_{p\,,\,r}$.

Here, q is the conjugate of p.

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