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Exponential - Logarithmic Series Fields .

Preliminaries.

Reference: *Ordered Exponential Fields*; The Fields Institute Monograph Series volume 12, AMS 2000.

Let $G \neq 1$ be an ordered abelian group.

• $\mathbb{R}((G))$ will denote the **field of generalized series** with real coefficients, of which support is an anti well ordered and subset of G.

• $f = \Sigma_{g \in G} f_g \, g$ with $f_g \in \mathbb{R}$ and

$$\text{supp}\,(f) := \{g \in G\,;\, f_g \text{ nonzero }\}$$

is and anti-wellordered.

• Pointwise addition, convolution formula for multiplication of series, anti-lexicographic order, natural valuation is given by "leading monomial".

• let $G^{\succ 1}$ be the semigroup of elements greater than 1.

• $\mathbb{R}((G^{\succ 1}))$ consists of "purely infinite" series with support in $G^{\succ 1}$.

• $\mathbb{R}((G^{\preceq 1}))$ and $\mathbb{R}((G^{\prec 1}))$ denote respectively the valuation ring of bounded elements, and the valuation ideal of infinitesimal elements of $\mathbb{R}((G))$.

We have the following direct sum (respectively, multiplicative direct sum) decompositions:

$$\mathbb{R}((G)) = \mathbb{R}((G^{\succ 1})) \oplus \mathbb{R} \oplus \mathbb{R}((G^{\prec 1})) \ , \tag{1}$$

$$\mathbb{R}((G))^{>0} = G \cdot \mathbb{R}^{>0} \cdot (1 + \mathbb{R}((G^{\prec 1}))) \ . \tag{2}$$

Indeed given $f \in \mathbb{R}((G))$ write

• $f = f^{\succ 1} + r + f^{\prec 1}$ and

• for $f > 0$ and $g :=$ max supp f, write

$$f = g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^{\prec 1}))$.

• If G is divisible, $\mathbb{R}((G))$ is a (non-archimedean) **real closed field**, i.e. by **Tarski's Tranfer Principle**, $\mathbb{R}((G))$ is elementarily equivalent to the ordered field of real numbers $(\mathbb{R}, <)$.

• What about $(\mathbb{R}, <, \exp)$?

• *How to construct nonarchimedean logarithmic fields using fields of generalized series?*

• The additive and multiplicative decompositions will be exploited.

• Use Taylor expansion of the logarithm to define the logarithm of a generalized series?

Summable families of series: Given a family

$$\{s_i\,; i \in I\} \subset \mathbb{R}((G))$$

how to make sense of $\sum_{1 \in I} s_i$ as an element of $\mathbb{R}((G))$?

• This is the case if (i) the support of the family, i.e. $\cup_{i \in I}$ support s_i is anti wellordered, and (ii) for every γ in the support of the family, the set of $i \in I$ for which $\gamma \in$ support s_i is *finite*.

• **B.H.Neumann:** For $\epsilon \in \mathbb{R}((G^{\prec 1}))$,

$$\sum_{i=1}^{+\infty} (-1)^{(i-1)} \frac{\epsilon^i}{i}$$

makes sense.

• The condition on ϵ is necessary!

Defining the logarithm.

• We have seen: the Taylor expansion defines a surjective logarithm from $\mathbb{R}^{>0} \cdot (1 + \mathbb{R}((G^{\prec 1})))$ onto $\mathbb{R} \oplus \mathbb{R}((G^{\prec 1}))$.

• A **logarithmic section** is an embedding of ordered groups

$$l : (G\,, \cdot, \prec) \to (\mathbb{R}((G^{\succ 1})), +)\ .$$

If we have a logarithmic section, we can define now a logarithm.

• Given $f \in \mathbb{R}((G))$, $f > 0$ and $g := \max \operatorname{supp} f$, write

$$f = g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^{\prec 1}))$.

• We extend l as follows:

$$L(f) = l(g \cdot c \cdot (1 + \epsilon)) = l(g) + \log c + \sum_{i=1}^{+\infty} (-1)^{(i-1)} \frac{\epsilon^i}{i}$$

• $L : (\mathbb{R}((G))^{>0}, \cdot) \rightarrow (\mathbb{R}((G)), +)$ is an order preserving embedding of groups, extending the logarithmic section l (the **logarithm** associated to the logarithmic section l).

Logarithmic sections from Hahn groups

Let us now consider a totally ordered set Γ,

• Consider the multiplicative "Hahn group" $H(\Gamma)$ which consists of formal products $g = \Pi f^r$, $f \in \Gamma$, $r \in \mathbb{R}$, with support g an anti well ordered subset of Γ. Multiplication is point wise, order is anti lexicographic, 1 is the product with empty support.

• Hahn Embedding's Theorem states that every ordered abelian group G is a subgroup of a Hahn group $H(\Gamma)$ (and Γ is uniquely determined by G).

• We shall from now on assume that G is a Hahn group $H(\Gamma)$, and explain how this data determines a logarithmic section:

• Consider $l : G \to \mathbb{R}((G^{\succ 1}))$ defined by

$$l(\prod f_i^{r_i}) := \sum r_i f_i \ ,$$

defines indeed a logarithmic section on $\mathbb{R}((G))$.

This logarithmic section has two defects:

(I) It violates the **growth axiom**.

(II) It does *not* map G **surjectively** onto the ring of purely infinite series $\mathbb{R}((G^{\succ 1}))$ (so its associated logarithm will not be surjective).

To construct *models* we shall fix these two defects as follows:

(I) We assume that Γ admits an order preserving automorphism which is a **leftward shift:**

$$\sigma(f) \prec f \text{ for all } f \in \Gamma \, .$$

- The automorphism σ induces the logarithmic section:

$$l(\prod f_i^{r_i}) := \sum r_i \sigma(f_i) \, .$$

This fixes (I) but is till not surjective. We shall now explain the core step in constructing exponentials of infinitely large elements to deal with (II): Since $l : G \to (\mathbb{R}((G^{\succ 1})), +)$ is not surjective, there exists elements of $\mathbb{R}((G^{\succ 1})) \setminus l(G)$ of which exponentials are not defined. We shall enlarge our group of monomials G to a group extension $G^{\#}$ to include the missing exponentials.

Exponential Extension

We take $G^{\#}$ to be a *multiplicative* copy $e[\mathbb{R}((G^{\succ 1}))]$ of $\mathbb{R}((G^{\succ 1}))$ over $l(G)$.

• More precisely, we construct $G^{\#}$ formally as follows:

$G^{\#} := \{e(\alpha); \alpha \in \mathbb{R}((G^{\succ 1})),\ \text{where } e(\alpha) := g \text{ if } \exists g \in G \text{ s.t. } \alpha = l(g)\}$

By its definition, G is a subset of $G^{\#}$.

• We define multiplication on $G^{\#}$ as follows:

$$e(\alpha_1)e(\alpha_2) := e(\alpha_1 + \alpha_2)\ .$$

In particular, if $g_1 = e(\alpha_1)$, $g_2 = e(\alpha_2) \in G$, then $e(\alpha_1)e(\alpha_2) = e(l(g_1) + l(g_2)) = e(l(g_1 g_2)) = g_1 g_2$, so

G is a *subgroup* of $G^{\#}$.

• We equip $G^{\#}$ with a total order:

$e(\alpha_1) < e(\alpha_2)$ if and only if $\alpha_1 < \alpha_2$ in $\mathbb{R}((G^{\succ 1}))$.

Again, if $g_1 = e(\alpha_1)$, $g_2 = e(\alpha_2) \in G$, then $e(\alpha_1) < e(\alpha_2)$ if and only if $l(g_1) < l(g_2)$ in $\mathbb{R}((G^{\succ 1}))$ if and only if $g_1 < g_2$ in G, so

G is an *ordered subgroup* of $G^{\#}$.

Since $G \subseteq G^{\#}$ as ordered abelian multiplicative groups, we view $\mathbb{R}((G))$ as an ordered subfield of $\mathbb{R}((G^{\#}))$ (by identifying $\mathbb{R}((G))$ with the elements of $\mathbb{R}((G^{\#}))$ having support in G).

• One verifies that the map

$$l^{\#} : (G^{\#}, \cdot) \to \mathbb{R}((G^{\# \succ 1})), +)$$

defined by:

$$l^{\#}(e(\alpha)) := \alpha$$

for $\alpha \in \mathbb{R}((G^{\succ 1}))$ is a prelogarithmic section with:

$$l^{\#}(G^{\#}) = \mathbb{R}((G^{\succ 1}))$$

and $l^{\#}$ extends l on G.

• By construction of the logarithms L and $L^{\#}$ on $\mathbb{R}((G))^{>0}$ and $\mathbb{R}((G^{\#}))^{>0}$ respectively, $L^{\#}$ is an extension of L.

We define the **exponential extension** of $(\mathbb{R}((G)), L)$ to be $(\mathbb{R}((G^{\#})), L^{\#})$.

The Exponential Closure

We now close under exponentiation by induction on n.

• If $n = 0$ set $(\mathbb{R}((G))^{\#n}, L^{\#n}) := (\mathbb{R}((G)), L)$.

For $n \in \mathbb{N}$, define inductively the *n-th exponential extension* of $(\mathbb{R}((G)), L)$:

$(\mathbb{R}((G))^{\#n}, L^{\#n}) :=$ the exponential extension of $(\mathbb{R}((G^{\#n-1})), L^{\#n-1})$.

• Set $\mathbb{R}((G))^{EL} := \cup\ \mathbb{R}((G))^{\#n}$ and $\mathrm{Log} := \cup\ L^{\#n}$.

We call $(\mathbb{R}((G))^{\mathrm{EL}}, Log)$ is EL-series field over (Γ, σ).

Rank and logarithmic rank

We see that pairwise distinct left shifts on Γ will induce pairwise distinct logarithms. We do more: we construct logarithms of pairwise distinct growth rates.

The **rank** of (Γ, σ) is the order type of the quotient $\Gamma / \sim_\sigma$, where $a \sim_\sigma a'$ if and only if there exists $n \in \mathbb{N}$ such that $\sigma^{(n)}(a) \geq a'$ and $\sigma^{(n)}(a') \geq a$.

Similarly the **logarithmic rank** of $(K^{>0}, l)$ is defined via the equivalence relation: a, $a' \in K^{>0}$ are *log-equivalent* if $a \sim_l a'$, that is, if and only if there exists

$$n \in \mathbb{N} \text{ such that } l^{(n)}(a) \leq a' \text{ and } l^{(n)}(a') \leq a\ .$$

Proposition 0.1 *The logarithmic rank of* $(\mathbb{R}((G)), l_\sigma)$ *is equal to the rank of* (Γ, σ).

An asymptotic scale indexed by $\aleph_1 \times \mathbb{Z}^2$.

We construct a totally ordered set of germs at infinity of real valued functions of a real variable, which admits $2^{\aleph_1}$ *left shifts.*

• For $(p, q) \in \mathbb{Z}^2$, we denote by $g_{p,q}$ the germ at $+\infty$ of the infinitely large *transmonomial*

$$x \mapsto \exp\left(x^q \exp\left(x^p\right)\right).$$

If we endow $\mathbb{Z}^2$ with the lexicographic order, then $(p, q) < (p', q')$ implies $g_{p,q} \prec g_{p',q'}$.

• Now let $\{h_\alpha\,; \alpha \in \aleph_1\}$ be a sequence of germs at $+\infty$ of infinitely large transmonomials h_α, in such a way that $\alpha < \beta$ implies $h_\alpha \prec h_\beta$.

• One can describe for example the first ϵ_0 terms of such a sequence. Set $h_0(x) := x$. We define h_α by transfinite induction for $\alpha < \epsilon_0$. If the Cantor normal form of α is $\omega^{\beta_r} d_r + \cdots + \omega^{\beta_1} d_1 + d_0$, with $\beta_1 < \cdots < \beta_r < \alpha$ and $d_{0,}, \ldots, d_r \in \mathbb{N}$, set

$$h_\alpha(x) := \exp\left(d_r h_{\beta_r}(x) + \cdots + d_1 h_{\beta_1}(x)\right) \exp(x)^{d_0} .$$

We can set $h_{\epsilon_0} := t(x)$ where $t(x)$ is a germ of transexponential growth.

• Finally: for all $(\alpha, p, q) \in \aleph_1 \times \mathbb{Z}^2$, we denote $f_{\alpha,p,q}$ the germ at $+\infty$ of the transmonomial $\exp_3(h_\alpha(x))\, g_{p,q}(x)$.

• These germs are defined in such a way that if $(\alpha, p, q) < (\alpha', p', q')$ for the lexicographic order, then $f_{\alpha,p,q} \prec f_{\alpha',p',q'}$. This set of germs Γ is thus totally ordered.

We construct $2^{\aleph_1}$ left-shifts of pairwise distinct ranks on Γ. To this end, we consider the two automorphisms defined on $\Gamma_1 = \left\{ g_{p,q} \, , \, (p,q) \in \mathbb{Z}^2 \right\}$ by :

$$\begin{aligned} \sigma\left(g_{p,q}\right) &= g_{p-1,q} \\ \rho\left(g_{p,q}\right) &= g_{p,q-1} \end{aligned}$$

It follows easily from the definition of $g_{p,q}$ that the rank of (Γ_1, σ) is 1 and the rank of (Γ_1, ρ) is $\mathbb{Z}$. We define now, for every $S \subset \aleph_1$, the decreasing automorphism τ_S on Γ by :

$$\tau_S\left(f_{\alpha,p,q}\right) = \begin{cases} f_{\alpha,p-1,q} = \exp_3\left(h_\alpha\right)\sigma\left(g_{p,q}\right) & \text{si } \alpha \in \mathrm{S} \\ f_{\alpha,p,q-1} = \exp_3\left(h_\alpha\right)\rho\left(g_{p,q}\right) & \text{si } \alpha \notin \mathrm{S} \end{cases}$$

The End