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Positivity in Power Series Rings.

Preliminaries.

- Let $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the ring of polynomials in n variables and real coefficients.

- A subset $M \subseteq \mathbb{R}[X]$ is a **quadratic module** if $1 \in M$, M is closed under addition and multiplication by squares (i.e. $a^2 f \in M$, $\forall a \in \mathbb{R}[X]$ and $f \in M$).

- A **quadratic preordering** is a quadratic module which is also closed under multiplication.

- The smallest preordering of $\mathbb{R}[X]$ is the set of **sums of squares** of $\mathbb{R}[X]$, denoted by $\sum \mathbb{R}[X]^2$.

• Given a finite subset $S = \{f_1, ..., f_s\}$ of $\mathbb{R}[X]$, the smallest preordering containing S (**preordering finitely generated by** S) is:

$$T_S = \{ \sum_{e \in \{0,1\}^s} \sigma_e f^e : \sigma_e \in \sum \mathbb{R}[X]^2, f_1, \cdots, f_s \in S \}$$

where $f^e := f_1^{e_1} \cdots f_r^{e_s}$, if $e = (e_1, \cdots, e_s)$.

• Let $S = \{f_1, \cdots, f_s\} \subset \mathbb{R}[X]$, S defines a **basic closed semialgebraic** subset of $\mathbb{R}^n$:

$$K = K_S = \{x \in \mathbb{R}^n \ : \ f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}$$

- Consider polynomials **positive semi-definite** on K_S:

$$\mathrm{Psd}(K_S) := \{f \in \mathbb{R}[X] : f(x) \geq 0 \text{ for all } x \in K_S\}$$

- $\mathrm{Psd}(K_S)$ is a preordering in $\mathbb{R}[X]$ and $T_S \subseteq \mathrm{Psd}(K_S)$.

Hilbert's 17th Problem is concerned with the issue of representation of positive semi-definite polynomials; motivated by the question: when it true that Psd$(K_S) = T_S$?

We say:

- T_S is **saturated** if $\mathrm{Psd}(K_S) = T_S$.

Low dimensional sets compact sets

In [S1] Scheiderer showed:

Theorem 0.1 *If dim(K_S) $\geq$ 3, then there exists a polynomial $p(X) \in \mathbb{R}[X]$ such that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ but $p \notin T_S$ (so T_S cannot be saturated).*

In the same paper, he also shows:

Theorem 0.2 *If $n = 2$ and K_S contains a cone of dimension 2, then there exists a polynomial $p(X) \in \mathbb{R}[X]$ such that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$ but $p \notin T_S$ (so T_S cannot be saturated).*

This left open the question formulated in [K-M]: Are there compact examples of $K_S \subseteq \mathbb{R}^2$ for which T_S is saturated?

Schmüdgen's Positivstellensatz and Scheiderer's Local Global Principle

In [Sc1] Schmüdgen proved the following result:

Theorem 0.3 *If K_S is compact, then $f > 0$ on K_S implies that $f \in T_S$.*

Note: the hypothesis $f > 0$ cannot be replaced by $f \geq 0$.

Recently, Scheiderer developed in a series of papers [S2], [S3], [S4] several *local global principles* to determine when a polynomial $f \geq 0$ on K_S belongs to T_S. His results generalize Schmüdgen's Striktpositivstellensatz.

Scheiderer's LGP states:

Theorem 0.4 *Suppose* $f, g_1, \ldots, g_s \in \mathbb{R}[X]$*, assume that* K_S *is compact,* $f \geq 0$ *on* K*, and* f *has just finitely many zeros in* K*. Then the following are equivalent:*

1. $f \in T_S$.
2. *For each zero* p *of* f *in* K*,* f *lies in the preordering of the completion of* $\mathbb{R}[X]$ *at* p *generated by* $g_1, \ldots, g_s$.

Where the **completion** of $\mathbb{R}[X]$ at $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$ is the ring of formal power series $\mathbb{R}[[X_1 - p_1, \cdots, X_n - p_n]]$.

Saturation for compact affine two dimensional sets.

In the two dimensional affine compact case his LGP allows him to show that certain finitely generated preorderings are saturated We need the following definitions to state the theorem. We say that p is a **singular** zero of g if

$$\delta g/\delta X(p) = \delta g/\delta Y(p) = 0 \; .$$

We say that g and h **meet transversally** at p if p is a non-singular zero of both g and h, and the curves $g = 0$ and $h = 0$ have distinct tangents at p.

Theorem 0.5 *Let $S = \{g_1, ..., g_s\}$ be irreducible polynomials in $\mathbb{R}[X, Y]$. Suppose that $K = K_S \subseteq \mathbb{R}^2$ is compact, and, for each boundary point p of K, either*

1. *there exists i such that p is a non-singular zero of g_i, and K is defined locally at p by the single inequality $g_i \geq 0$; or*

2. *there exists i, j such that p is a non-singular zero of g_i and g_j, g_i and g_j meet transversally at p, and K is defined locally at p by $g_i \geq 0$, $g_j \geq 0$.*

Then the preordering of $\mathbb{R}[X, Y]$ generated by $g_1, \ldots, g_s$ is saturated.

With these tools, he was able to produce examples:

Example 0.6

- The preordering generated by:
$S = \{1+X, 1-X, 1+Y, 1-Y\}$ (compact unit square K_S) is saturated.
- The preordering generated by:
$S = \{1-X^2-Y^2,\}$ (compact unit circle K_S) is saturated.
- $S = \{X^3-Y^2, 1-X\}$ then $X \geq 0$ on K_S but $X \notin T_S$ ($(0,0)$ is a singular zero of X^3-Y^2).
- $S = \{Y, Y-X^3, 1-X\}$, $X \geq 0$ on K_S but $X \notin T_S$ (the curves $Y=0$ and $Y=X^3$ do not meet transversally at $(0,0)$).

In [CKM] we extend Scheiderer's theorem to handle certain cases where the boundary curves do not meet transversally, for example our theorem covers the following examples:

Example 0.7 $S = \{X, 1-X, Y, X^2-Y\}$ or $S = \{1+X, 1-X, Y, X^2-Y\}$. In these examples, the boundary curves $Y = 0$ and $Y = X^2$ share a common tangent at the origin, so Scheiderer's does not apply. The fact that saturation holds in these examples is a consequence of our main result, Corollary 0.8 in the next slide.

CKM Generalization of Scheiderer's Theorem.

Corollary 0.8 *Let* $S = \{g_1, ..., g_s\}$ *be irreducible polynomials in* $\mathbb{R}[X, Y]$*. Suppose that* $K = K_S \subseteq \mathbb{R}^2$ *is compact, and, for each boundary point* p *of* K*, either*

1. *there exists* i *such that* p *is a non-singular zero of* g_i*, and* K *is defined locally at* p *by the single inequality* $g_i \geq 0$*; or*

2. *there exists* i, j *such that* p *is a non-singular zero of* g_i *and* g_j*,* g_i *and* g_j *meet transversally at* p*, and* K *is defined locally at* p *by* $g_i \geq 0$*,* $g_j \geq 0$*; or*

3. *there exists i, j such that p is a non-singular zero of g_i and g_j, g_i and g_j share a common tangent at p but do not cross each other at p, and K is described locally at p as the region between $g_i = 0$ and $g_j = 0$; or*

4. *there exists i, j, k such that p is a non-singular zero of g_i, g_j and g_k, g_i and g_j share a common tangent at p, g_i and g_k meet transversally at p, and K is described locally at p as the part of the region between $g_i = 0$ and $g_j = 0$ defined by $g_k \geq 0$.*

Then the preordering of $\mathbb{R}[X, Y]$ generated by $g_1, \ldots, g_s$ is saturated.

The End

References

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