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Real closed fields and fragments of Peano arithmetic

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I. Integer parts of totally ordered fields and models of open induction.

In this talk “order” means **total** order.

(i) An **integer part (IP)** Z of an ordered field K is a discretely ordered subring (1 is least positive element) such that

$$\forall x \in K \, \exists z \in Z : \; z \leq x < z + 1 \, .$$

$z := \lfloor x \rfloor$ Gauß bracket.

(ii) **Peano arithmetic (PA)**

is the first-order theory, in the language $L := \{+, \cdot, <, 0, 1\}$, of discretely ordered commutative rings with 1 whose set of non-negative elements satisfies, for each formula $\Phi(x, \overline{y})$, the associated induction axiom:

$$\forall \overline{y} \left[\Phi(0, \overline{y}) \& \forall x \left[\Phi(x, \overline{y}) \rightarrow \Phi(x+1, \overline{y}) \right] \rightarrow \forall x \Phi(x, \overline{y}) \right].$$

(iii) K is **real closed** if every positive element has a square root in K, and every polynomial in $K[x]$ of odd degree has a root in K.

Tarski(1931) K is real closed iff K is elementarily equivalent to $(\mathbb{R}, +, \cdot, 0, 1, <)$

(iv) **Open Induction (OI)** is the fragment of PA obtained by taking the induction axioms associated to open formulas only.

Theorem [Shepherdson]

IP's of real closed fields are precisely the models of OI.

We shall exploit this correspondence to construct non-standard models of fragments of arithmetic ...

Remarks:

• $\mathbb{Z}$ is an IP of K iff K is archimedean ([Hölder] iff K is isomorphic to a subfield of $\mathbb{R}$).

We will only consider **non-archimedean** fields.

• An ordered field K need not admit an IP.

• In general, different IP's need not be isomorphic, not even elementarily equivalent.

Does every real closed field admit an IP?

If yes, how to construct such?

Let us first construct real closed fields...

II. Algebraic constructions.

(DOAG) Divisible ordered abelian groups:

• Let Γ be any ordered set, $\{A_\gamma ; \gamma \in \Gamma\}$ a family of divisible archimedean groups (subgroups of $\mathbb{R}$).

• For $g \in \Pi_\Gamma A_\gamma$, set

$$\text{support} g := \{\gamma \in \Gamma \, ; g_\gamma \neq 0\}$$

• The **Hahn group** is the subgroup of $\Pi_\Gamma A_\gamma$

$$\mathbf{H}_\Gamma A_\gamma := \{g \, ; \, \text{support} g \text{ is well-ordered in } \Gamma\}$$

ordered lexicographically by "first differences".

• The **Hahn sum** is the subgroup

$$\oplus_\Gamma A_\gamma := \{g \, ; \, \text{support} g \text{ is finite}\}$$

Theorem [Hahn's embedding Theorem (1907)]

Let G be a divisible ordered abelian group, with rank Γ and archimedean components $\{A_\gamma\,;\,\gamma \in \Gamma\}$. Then G is (isomorphic to) a subgroup of $\mathbf{H}_\Gamma A_\gamma$.

Above, the rank and archimedean components are valuation theoretic invariants of G and can be described explicitly $\cdots$

(RCF) Real closed fields:

• Let G be any divisible ordered abelian group, k a real closed archimedean field (a real closed subfield of $\mathbb{R}$).

• The **Hahn field** is the field of generalized power series

$$k((G)) = \{s = \sum_{g \in G} s_g t^g ; \text{support} s \text{ is well-ordered in } G\}$$

with convolution multiplication (Cauchy product) and lexicographic order.

The Hahn field $\mathbb{K}$ is a **valued** field: the map

$$v : \mathbb{K} \to G \cup \infty$$

$$v(s) := \min \text{support} s$$

is a **valuation** with

- **valuation ring** $\mathcal{O} := k((G^{\geq 0}))$
- **group of units** $\mathcal{O}^{\times}$
- **valuation ideal** $\mathcal{M} := k((G^{+}))$
- **residue field** k
- **value group** G.

Theorem [Kaplansky Embedding's Theorem (1942)]: Let K be real closed field with residue field k and value group G. Then K is (analytically isomorphic to) a subfield of a field of $k((G))$.

Here, we mean the natural valuation, with valuation ring the convex hull of $\mathbb{Z}$ in K.

So we know how to construct all DOAG and all RCF, now we want to contruct IP of RCF ...

III. Truncation Integer Parts

Direct sum (respectively product) decompositions:

$$\begin{array}{rcccccc} k((G)) & = & k((G^-)) & \oplus & k & \oplus & k((G^+)) \\ k((G))^{>0} & = & t^G & \times & k^+ & \times & [\,1 + k((G^+))\,] \end{array}$$

Indeed given $s \in k((G))$ write

- $s = s_{<0} + s_0 + s_{>0}$ and
- for $s > 0$ and $g = v(s) = \min \text{support} s$, write

$$s = t^g \cdot s_g \cdot (1 + \epsilon)$$

with $s_g \in k^+$, $\epsilon \in k((G^+))$.

Proposition

$Z := k((G^-)) \oplus \mathbb{Z}$ is an IP of $\mathbb{K}$.

Proof: Clearly, Z is a discrete subring. Let $s \in k((G))$. Let $\lfloor s_0 \rfloor \in \mathbb{Z}$ be the integer part of $s_0 \in k$. Define

$$z_s = \begin{cases} s_{<0} + s_0 - 1 & \text{if } s_0 \in \mathbb{Z} \text{ and } s_{>0} < 0, \\ s_{<0} + \lfloor s_0 \rfloor & \text{otherwise.} \end{cases}$$

Clearly, $z_s \leq s < z_s + 1$.

Observation: If F is a **truncation closed** subfield of $\mathbb{K}$ ($\forall s : s \in F$ implies $s_{<0} \in F$), then $Z_F := [k((G^-)) \cap F] \oplus \mathbb{Z}$ is an IP of F.

[Mourgues-Ressayre or Kaplansky revisited]

Let K be real closed field with residue field k and value group G. Then K is (isomorphic to) a truncation closed subfield of a field of $k((G))$, thus K has an IP.

TIP • need not have cofinal set of primes.

• they are never normal

• they are never models of PA.

Does a RCF admit an IP which is a model of normal open induction? of full PA?

First is still open, we now answer second...the key is...

IV. (IPA) Integer Parts that are models of PA

FACT: [Exponentiation on the non-negative elements of a model of PA] The graph of the exponential function $2^y = z$ on $\mathbb{N}$ is definable by an L-formula, and PA proves the basic properties of exponentiation. Thus any model of PA is endowed with an **exponential function** exp.

As observed by D. Marker, this provides a key connection to real closed exponential fields which we shall now explain and exploit...

Direct sum (respectively product) decompositions hold for any RCF K with valuation ring $\mathcal{O}$, value group G and residue field $\overline{K}$:

$$(K, +) = \mathbb{A} \oplus \mathcal{O}$$

$$(K^+, \cdot) = \mathbb{B} \times \mathcal{O}_+^\times$$

where $\mathbb{A}$ and $\mathbb{B}$ are unique up to isomorphism, the rank of $\mathbb{A}$ is (isomorphic to) G^-, its archimedean components are (isomorphic to) $\overline{K}$ and $\mathbb{B} \simeq G$.

A RCF K has **left exponentiation** iff there is an isomorphism from a group complement $\mathbb{A}$ of $\mathcal{O}$ in $(K, +, 0, <)$ onto a group complement $\mathbb{B}$ of $\mathcal{O}_+^\times$ in $(K^+, \cdot, 1, <)$.

Let G be a DOAG with rank Γ and archimedean components $\{A_\gamma : \gamma \in \Gamma\}$. We say that G is an **exponential group** (in C) if Γ is isomorphic (as linear order) to the negative cone G^-, and each A_γ is isomorphic (as ordered group) to C, for some archimedean group C.

[Characterization of countable exponential groups (S.K)]

A countable DOAG $G \neq 0$ is an exponential group if and only if G is isomorphic to the Hahn sum $\oplus_{\mathbb{Q}} C$ for some countable archimedean group $C \neq 0$.

Theorem:[S.K]

If K admits a left exponential, then the value group G of K is an exponential group in $\overline{K}$.

Corollary[Carl-D'Aquino-S.K]

If K admits an IPA (i.e K is an IPA real closed field), then K admits a left exponential, therefore the value group of K is an exponential group in $\overline{K}$.

Remark:

There are plenty of DOAG that are not exponential groups in $\overline{K}$. For example, take the Hahn group $G = \mathbf{H}_{\gamma\in\Gamma} A_\gamma$ where the archimedean components A_γ are divisible but not all isomorphic and/or Γ is not a dense linear order without endpoints (say, a finite Γ). Alternatively, we could choose all archimedean components to be divisible and all isomorphic, say to C, and Γ to be a dense linear order without endpoints, but choose the residue field so that $\overline{K}$ not isomorphic to C.

A class of not IPA real closed fields.

Let k be any real closed subfield of $\mathbb{R}$. Let $G \neq \{0\}$ be any DOAG which is *not* an exponential group in k. Consider the Hahn field $k((G))$ and its subfield $k(G)$ generated by k and $\{t^g : g \in G\}$. Let K be any real closed field satisfying

$$k(G)^{rc} \subseteq K \subseteq k((G))$$

where $k(G)^{rc}$ is the real closure of $k(G)$. Any such K has G as value group and k as residue field. By Corollary above, K does not admit an IPA.

IPA real closed fields are recursively saturated, and the converse holds in the countable case [e.g. Theorem 5.1 and 5.2 of *Real closed fields and models of Peano arithmetic*; D'Aquino-Knight -Starchenko (JSL 2010)].

We now relate IPA real closed fields to the algebraic characterization of recursive saturation given in D'Aquino-S.K-Lange

V. Recursively Saturated real closed fields.

Theorem [Valuation theoretic characterization of recursively saturated real closed fields](D'Aquino-S.K-Lange)

If R is a real closed field with natural valuation v, value group G. Then R is recursively saturated in the language of ordered rings if and only if

1. The residue field is a Scott set S
2. G is recursively saturated with Archimedean components all equal to S;
3. every pseudo Cauchy sequence of length ω that is computable in an element of S over some finite tuple of parameters in R has a pseudo limit in R; and
4. every type realized by some finite tuple $\bar{a}$ in R is computable in an element of S.

Therefore a countable field is an IPA real closed field if and only if the above 4 conditions hold.

Moreover, the condition on the value group is explicit via the following:

Theorem [Valuation theoretic characterization of recursively saturated divisible ordered abelian groups] (Harnik-Ressayre and D'Aquino-S.K-Lange)

Let $G \neq 0$ be a DOAG. Then G is recursively saturated in the language of ordered groups if and only if

1. the value set Γ of G is a dense linear order without endpoints, and
2. all Archimedean components of G equal a common Scott set S.

And again the countable case is special...

[Characterization of countable recursively saturated DOAG](D-K-L)

A countable DOAG $G \neq 0$ is recursively saturated if and only if G is isomorphic to $\oplus_{\mathbb{Q}} S$ for some countable Scott set S.

We conclude: if R is an IPA real closed field then on the one hand the value group is exponential in the residue field, and on the other hand the value group is recursively saturated.

This begs the question:

what is the relationship between these 2 classes of groups?

In general there are no implications either way. However, comparing both characterizations in the countable case, we clearly see that recursively saturated DOAG implies exponential group:

Corollary: a countable exponential group in C is recursively saturated if and only if C is a countable Scott set.

Definitions:

Let • L be a computable language. An L-structure M is **recursively saturated** if for every computable set of L-formulas $\tau(x, \bar{y})$ and every tuple $\bar{a}$ in M (of the same length as $\bar{y}$) such that $\tau(x, \bar{a})$ is finitely satisfiable in M, then $\tau(x, \bar{a})$ is realized in M.

• A subset $\mathcal{T} \subset 2^{<\omega}$ is a **tree** if every substring of an element of $\mathcal{T}$ is also an element of $\mathcal{T}$. If $\sigma, \tau \in 2^{<\omega}$, we let $\sigma \prec \tau$ denote that σ is a substring of τ. A sequence $f \in 2^{\omega}$ is a **path** through a tree $\mathcal{T}$ if for all $\sigma \in 2^{<\omega}$ with $\sigma \prec f$, we have $\sigma \in \mathcal{T}$. For any $\sigma \in 2^{<\omega}$, the length of σ, denoted by $length(\sigma)$, is the unique $n \in \omega$ satisfying $\sigma \in 2^{n}$.

• A nonempty set $S \subset \mathbb{R}$ is a **Scott set** if

1. S is **computably closed**, i.e., if $r_1, \ldots r_n \in S$ and $r \in \mathbb{R}$ is computable from $r_1 \oplus \ldots \oplus r_n$ (the *Turing join* of $r_1, \ldots, r_n$), then $r \in S$.
2. If an infinite tree $\mathcal{T} \subset 2^{<\omega}$ is computable in some $r \in S$, then $\mathcal{T}$ has a path that is computable in some $r' \in S$.

Fact A Scott set is an Archimedean real closed field.

• Let λ be an infinite ordinal. A sequence $(a_\rho)_{\rho<\lambda}$ is **pseudo Cauchy** if for every $\rho < \sigma < \tau < \lambda$ we have $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$. We say that x is a **pseudo limit** of the pseudo Cauchy sequence $(a_\rho)_{\rho<\lambda}$ if $v(x - a_\rho) = v(a_{\rho+1} - a_\rho)$ for all $\rho < \lambda$.

The End

References

[1] D. Biljakovic, M. Kotchetov, S. Kuhlmann, *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields*, Logic, Algebra and Arithmetic, LNL **26**, A. Symbolic Logic, 42-65 (2006)

[2] M. Carl, P. D'Aquino, S. Kuhlmann, *Value groups of real closed fields and fragments of Peano arithmetic* (arXiv: 1205.2254) (2012)

[3] P. D'Aquino, J.F. Knight, S. Kuhlmann and K. Lange, *Real closed exponential fields*, Fund. Math. 219, 163-190 (2012)

[4] P. Hajek and P. Pudlak, *Metamathematics of First-Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, (1988).

[5] S. Kuhlmann, *Ordered Exponential Fields*, Fields Institute Monographs **12**, (2000).

[6] M.-H. Mourgues and J.- P. Ressayre, *Every real closed field has an integer part*, J. Symbolic Logic, 58, 641–647 (1993).

[7] J. C. Shepherdson, *A non-standard model for a free variable fragment of number theory*, in Bull. Acad. Polonaise des Sciences 12, 79-86 (1964).