AMS.MAA Joint Mathematics Meetings, Seattle JAN 6-9, 2016

January 8, 2016

Salma Kuhlmann [1]
Schwerpunkt Reelle Algebra und Geometrie,
Fachbereich Mathematik und Statistik,
Universität Konstanz,
78457 Konstanz, Germany

Email: salma.kuhlmann@uni-konstanz.de

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http://www.math.uni-konstanz.de/ kuhlmann/vortraege.htm

[1]Supported by Ausschuß für Forschungsfragen der Universität Konstanz

Exponential - logarithmic power series fields and the surreal numbers.

The aim of this talk is to survey some of our work on exponential-logarithmic power series fields relevant to the surreals, which lead us to

our discovery of the class of *kappa-surreal numbers* and our formulation of three conjectures:

Conjecture 1: the kappa-surreal numbers generate the chain of *log-atomic surreal numbers*,

Conjecture 2: the field of surreal numbers No is an exponential- logarithmic power series field over this chain of *initial fundamental monomials*,

Conjecture 3: No can therefore be equipped with a surjective derivation which makes it into a *universal domain for ordered differential fields of Hardy type* .

Part I: The κ-bounded EL- series field $\mathbf{EL}(\Gamma, \sigma)$:

Constructing models of real exponentiation starting with a chain of initial fundamental monomials Γ equipped with a right shift automorphism σ, highlighting exponential rank.

• Kuhlmann, F.- V. - Kuhlmann, S.: *Explicit construction of exponential-logarithmic power series*, Prépublications de Paris 7 61, (1995-1996).

• Kuhlmann : *Ordered Exponential Fields*, Fields Institute Monograph Series vol. 12, AMS (2000)

• Kuhlmann- Shelah: *κ–bounded Exponential Logarithmic Power Series Fields*, Annals for Pure and Applied Logic, 136, 284-296, (2005).

• Kuhlmann- Tressl: *Comparison of Exponential-Logarithmic and Logarithmic-Exponential series* Math. Logic Quarterly, 58, 434-448 (2012)

Part II: Hardy Type Derivations on EL- series and applications to Schanuel's conjecture:

Developing a general method to construct derivations on $\mathrm{EL}(\Gamma, \sigma)$ starting with an arbitrary definition of a derivation of the initial fundamental monomials. Combinatorial, set theoretic criteria on supports.

- Kuhlmann- Matusinski: *Hardy type derivations in generalized series fields*, J. of Algebra, 351, 185-203, (2012)
- Kuhlmann- Matusinski: *Hardy type derivations on fields of exponential logarithmic series*, J. of Algebra, 345, 171-189 (2011)
- Kuhlmann- Matusinski- Shkop: *A Note on Schanuel's Conjectures for Exponential Logarithmic Power Series Fields*, Archiv der Mathematik, 100, 431-436 (2013)

Part III: The exponential rank of Gonshors exponentiation on the surreals.

Investigating the log-exp equivalence classes and fixing a canonical system of representatives, the class of kappa numbers, which contains strictly the class of generalised epsilon numbers and is strictly contained in the class of log-atomic surreals.

- Kuhlmann- Matusinski: *The exponential-logarithmic equivalence classes of surreal numbers*, ORDER - A Journal on the Theory of Ordered Sets and its Applications, 32, 53-68 (2014)

Part IV: Integer parts of fields of power series, models of arithmetic and the ring of omnific surreals:

- Biljakovic- Kotchetov-Kuhlmann: *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields*, LNL 26, Association for Symbolic Logic, 42-65 (2006)
- Fornasiero- Kuhlmann, F.-V.- Kuhlmann, S.: *Towers of complements and truncation closed embeddings of valued fields* J. of Algebra, 323, 574-600 (2010)

PART I

The natural valuation.

• Let G be a totally ordered abelian group. The archimedean equivalence relation on G is defined as follows. For $0 \neq x$, $0 \neq y \in G$:

$$x \overset{+}{\sim} y \text{ if } \exists n \in \mathbb{N} \text{ s.t. } n|x| \geq |y| \text{ and } n|y| \geq |x|$$

where $|x| := \max\{x, -x\}$. We set $x << y$ if for all $n \in \mathbb{N}$, $n|x| < |y|$. We denote by $[x]$ is the archimedean equivalence class of x. We totally order the set of archimedean classes as follows: $[y] < [x]$ if $x << y$.

• Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. Using the archimedean equivalence relation on the ordered abelian group $(K, +, 0, <)$, we can endow K with the **natural valuation** v:

for $x, y \in K, x, y \neq 0$ define $v(x) := [x]$ and $[x]+[y] := [xy]$.

Notation:

Value group: $v(K) := \{v(x) \mid x \in K, x \neq 0\}$.
Valuation ring:, $R_v := \{x \mid x \in K \text{ and } v(x) \geq 0\}$.
Valuation ideal: $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$.
Group of positive units:
$U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$.

Ordered Exponential Fields.

An ordered field K is an **exponential field** if there exists a map

$$\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$$

such that exp is an isomorphism of ordered groups. A map exp with these properties will be called an **exponential** on K. A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. We require the exponentials (logarithms) to be v**-compatible**:

$$\exp(R_v) = U_v^{>0}.$$

We are interested in exponentials (logarithms) satisfying the **growth axiom** scheme: **(GA)**:

$$\forall n \in \mathbb{N} : \; x > \log(x^n) = n\log(x) \text{ for all } x \in K^{>0} \setminus R_v \; .$$

Via the natural valuation v, this is equivalent to

$$v(x) < v(\log(x)) \text{ for all } x \in K^{>0} \setminus R_v \; . \tag{1}$$

A logarithm log is a **(GA)-logarithm** if it satisfies (1).

Hahn Groups and Fields.

• Let Γ be any totally ordered set and R any ordered abelian group. Then R^Γ is the set of all maps g from Γ to R such that the **support** $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ. Endowed with the lexicographic order and pointwise addition, R^Γ is an ordered abelian group, called the **Hahn group**.

• **Representation for the elements of Hahn groups:** Fix a strictly positive element $\mathbf{1} \in R$ (if R is a field, we take $\mathbf{1}$ to be the neutral element for multiplication). For every $\gamma \in \Gamma$, we will denote by $\mathbf{1}_\gamma$ the map which sends γ to $\mathbf{1}$ and every other element to 0 ($\mathbf{1}_\gamma$ is the characteristic function of the singleton $\{\gamma\}$.) For $g \in R^\Gamma$ write

$$g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$$

(where $g_\gamma := g(\gamma) \in R$).

• For $G \neq 0$ an ordered abelian group, k an archimedean ordered field, $k((G))$ is the (generalized) **power series field** with coefficients in k and exponents in G. As an ordered abelian group, this is just the Hahn group k^G. A series $s \in k((G))$ is written

$$s = \sum_{g \in G} s_g t^g$$

with $s_g \in k$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$.

• The natural valuation on $k((G))$ is $v(s) = \min \operatorname{support} s$ for any series $s \in k((G))$. The value group is G and the residue field is k. The valuation ring $k((G^{\geq 0}))$ consists of the series with non-negative exponents, and the valuation ideal $k((G^{>0}))$ of the series with positive exponents. The **constant term** of a series s is the coefficient s_0. The units of $k((G^{\geq 0}))$ are the series in $k((G^{\geq 0}))$ with a non-zero constant term.

• **Additive Decomposition** Given $s \in k((G))$, we can truncate it at its constant term and write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus a complement in $(k((G)), +)$ to the valuation ring is the Hahn group $k^{G^{<0}}$. We call it the **canonical complement to the valuation ring** and denote it by $k((G^{<0}))$.

• **Multiplicative Decomposition** Given $s \in k((G))^{>0}$, we can factor out the monomial of smallest exponent $g \in G$ and write $s = t^g u$ with u a unit with a positive constant term. Thus a complement in $(k((G))^{>0}, \cdot)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g. We call it the **canonical complement to the positive units** and denote it by **Mon** $k((G))$.

κ–bounded Hahn Groups and Fields.

Fix a regular uncountable cardinal κ.

• The **κ-bounded Hahn group** $(R^\Gamma)_\kappa \subseteq R^\Gamma$ consists of all maps of which support has cardinality $< \kappa$.

• The **κ-bounded power series field** $k((G))_\kappa \subseteq k((G))$ consists of all series of which support has cardinality $< \kappa$. It is a valued subfield of $k((G))$. We denote by $k((G^{\geq 0}))_\kappa$ its valuation ring. Note that $k((G))_\kappa$ contains the monic monomials. We denote by $k((G^{<0}))_\kappa$ the complement to $k((G^{\geq 0}))_\kappa$.

• **Our first goal** is to define an exponential (logarithm) on $k((G))_\kappa$ (for appropriate choice of G). From the above discussion, we get:

Proposition 0.1 *Set $K = k((G))_\kappa$. Then $(K, +, 0, <)$ decomposes lexicographically as the sum:*

$$(K, +, 0, <) = k((G^{<0}))_\kappa \oplus k((G^{\geq 0}))_\kappa \ . \qquad (2)$$

$(K^{>0}, \cdot, 1, <)$ decomposes lexicographically as the product:

$$(K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0} \qquad (3)$$

Proposition 0.1 allows us to achieve our goal in two main steps; by defining the logarithm on $\mathrm{Mon}\,(K)$ and on $U_v^{>0}$.

The Main Step

Theorem 0.2 *Let* Γ *be a chain,* $G = (\mathbb{R}^\Gamma)_\kappa$ *and* $K = \mathbb{R}((G))_\kappa$*. Assume that*

$$l : \Gamma \to G^{<0}$$

is an isomorphism of chains. Then l *induces an isomorphism of ordered groups (a logarithm)*

$$log\ : (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$$

as follows: given $a \in K^{>0}$*, write* $a = t^g r(1+\varepsilon)$ *(with* $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$*,* $r \in \mathbb{R}^{>0}$*,* ε *infinitesimal), and set*

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i} \tag{4}$$

This logarithm satisfies

$$v(\log t^g) = l(\min \operatorname{support} g) \tag{5}$$

Moreover, log *satisfies* **GA** *if and only if*

$$l(\min \operatorname{support} g) > g \qquad \text{for all } g \in G^{<0}\ . \tag{6}$$

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Getting such isomorphisms l:

Let Γ be any chain, $G = (\mathbb{R}^\Gamma)_\kappa$ and $K = \mathbb{R}((G))_\kappa$. Then

$$\iota : \Gamma \to G^{<0} \ \text{ defined by } \gamma \mapsto -\mathbf{1}_\gamma$$

is an **embedding** of chains, and gives rise to prelogaritm on K. However, this prelogarithm is neither surjective nor does it satisfy **GA**.

- To get a prelogarithm satisfying **GA**, we choose $\sigma \in \mathrm{Aut}\,(\Gamma)$ with the property that

$$\sigma(\gamma) > \gamma \text{ for all } \gamma \in \Gamma \tag{7}$$

(We say that σ is an **increasing** automorphism). We set $l = \iota \circ \sigma$. Now

$$l : \Gamma \to G^{<0} \text{ defined by } \gamma \mapsto -\mathbf{1}_\sigma(\gamma)$$

is an embedding of chains satisfying (6), so gives rise to a prelogaritm on K satisfying **GA**-

- To get a surjective prelogarithm, we have to modify Γ as follows:

Proposition 0.3 *Let $\Gamma \neq \emptyset$ be a given chain. There is a canonically constructed chain $\Gamma_\kappa \supseteq \Gamma$ together with an* **isomorphism** *of ordered chains*

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

where $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$. Moreover, every increasing $\sigma \in$ Aut (Γ) extends canonically to an increasing $\sigma_\kappa \in$ Aut (Γ_κ)

We call the pair $(\Gamma_\kappa, \iota_\kappa)$ the κ-th **iterated lexicographic power** of Γ.

Procedure of constructing the **Exponential-Logarithmic field of κ-bounded series over** (Γ, σ): Let Γ be given and σ an increasing automorphism.

- Define Γ_κ, G_κ, ι_κ, and σ_κ as above.

- Set $K := \mathbb{R}((G_\kappa))_\kappa$ and $l := \iota_\kappa \circ \sigma_\kappa$. For $a \in K^{>0}$ write $a = t^g r(1+\varepsilon)$ where $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$, $r \in \mathbb{R}^{>0}$, ε infinitesimal, then

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{-\mathbf{1}_{\sigma_\kappa(\gamma)}} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i} \quad (8)$$

- Set $\exp = \log^{-1}$.

- $(K, \exp)$ is a model of $T_{\mathrm{an,exp}}$.

Exponential-Logarithmic Equivalence.

• Let Γ be a chain and $\sigma \in \mathrm{Aut}(\Gamma)$ an increasing automorphism. By induction, we define the **n-th iterate** of σ: $\sigma^1(\gamma) := \sigma(\gamma)$ and $\sigma^{n+1}(\gamma) := \sigma(\sigma^n(\gamma))$. Define an equivalence relation on Γ as follows: For $\gamma, \gamma' \in \Gamma$, set

$$\gamma \sim_\sigma \gamma' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \sigma^n(\gamma) \geq \gamma' \text{ and } \sigma^n(\gamma') \geq \gamma \,.$$

The equivalence classes $[\gamma]_\sigma$ of $\sim_\sigma$ are convex and closed under application of σ (they are the convex hulls of the orbits of σ). The order of Γ induces an order on $\Gamma/\sim_\sigma$. The order type of $\Gamma/\sim_\sigma$ is the **rank** of (Γ, σ).

Example 0.4 Let $\Gamma = \mathbb{Z} \vec{\coprod} \mathbb{Z}$ (i.e. the lexicographically ordered Cartesian product $\mathbb{Z} \times \mathbb{Z}$) endowed with the automorphism $\sigma((x, y)) := (x, y + 1)$. The rank of σ is $\mathbb{Z}$. Now consider the increasing automorphism
$\tau((x, y)) := (x + 1, y)$. The rank of τ is 1.

• Let K be a real closed field and log a (**GA**)- logarithm on $K^{>0}$. Define an equivalence relation on $K^{>0} \setminus R_v$:

$$a \sim_{log} a' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \log_n(a) \leq (a') \text{ and } \log_n(a') \leq a$$

(where $\log_n$ is the n-th iterate of the log). The order type of the chain of equivalence classes is the **logarithmic rank** of $(K^{>0}, \log)$.

We can compute the logarithmic rank of the Exponential-Logarithmic field of κ-bounded series over (Γ, σ):

Theorem 0.5 *The logarithmic rank of* $(\mathbb{R}((G_\kappa))_\kappa^{>0}, \log)$ *is equal to the rank of* (Γ, σ).

This proof (as many other proofs) is based on the observation that every series is log-equivalent to a **fundamental monomial**, that is a monomial of the form

$$t^{-\mathbf{1}_\gamma} \text{ with } \gamma \in \Gamma\ .$$

Next one observes that

$$\text{for all } \gamma, \gamma' \in \Gamma : t^{-\mathbf{1}_\gamma} \sim_{log} t^{-\mathbf{1}_{\gamma'}} \text{ if and only if } \gamma \sim_\sigma \gamma'\ .$$

This in turn is based on the following useful formula for $\log_n(t^{-\mathbf{1}_\gamma})$: by induction,

$$\log{}_n(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma^n(\gamma)}}\ .$$

Remark 0.6 If Γ admits automorphisms of distinct rank, then $(\mathbb{R}((G_\kappa))$ admits logarithms of distinct logarithmic rank. We can also use this observation to introduce **transexponentials**, as illustrated in the next example.

Example 0.7 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$, $\sigma((x,y)) := (x, y+1)$, $(K, \log)$ the corresponding κ-bounded model. For the automorphism $\tau((x,y)) := (x+1, y)$, let L, respectively $T := L^{-1}$ be the corresponding induced logarithm and exponential on K.

Effect of σ, τ on the fundamental monomials: let $\gamma = (x,y) \in \Gamma$, then

$$\log(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma(\gamma)}} ,$$

Whereas

$$L(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\tau(\gamma)}} ,$$

We see that, for any fundamental monomial $X := t^{-\mathbf{1}_\gamma}$ and any $n \in \mathbb{N}$ we have:

$$L(X) < \log_n(X) .$$

Also, a simple computation (using the fact that σ and τ commute) shows that also, for all $n \in \mathbb{N}$:

$$T(X) > \exp_n(X) .$$

Remark: Note that $\mathbf{L} := \sum \log_n X$ is a well-defined element of $\mathrm{EL}_\kappa(\Gamma, \sigma)$, this is the reason that this field cannot be isomorphic to a field of Transseries (result with Tressl).

In Part II, we see how the logarithm determines the derivation. We obtain fields equipped with widely distinct derivations.

PART II

Main Motivation: We want a "Kaplansky embedding Theorem" for ordered differential fields. The κ-bounded fields of power series are good candidates as "universal domains". But for this to make sense, we need first had to endow them with a good differential structure.

Derivations: We want to endow the field of κ-bounded series over (Γ, σ) with a derivation d satisfying the following properties:

• d is strongly linear, that is

$$d\sum_g r_g t^g \ = \ \sum_g r_g dt^g \ . \tag{9}$$

• d satisfies strong Leibniz rule:

$$d(t^g) = d(\prod t_\gamma^{g_\gamma}) = t^g \sum g_\gamma (d(\gamma)/\gamma) \tag{10}$$

where $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$ and $t_\gamma := t^{-\mathbf{1}_\gamma}$ here and below to simplify notations.

• d satisfies the rule for the logarithmic derivative:

$$d \log a = da/a, \ \text{for } a > 0. \tag{11}$$

Reductions: The above rules direct us to perform a number of steps in trying to define derivatives:

(i) From (9) and (10), it is clear that we need to determine dt^g, for $g \in G^{<0}$.

(ii) From (11) determining dt^g reduces to determining $d \log t^g$.

(iii) By definition of log, this in turn reduces to determining $d \log t^{-\mathbf{1}_\gamma}$, for a fundamental monomial $t^{-\mathbf{1}_\gamma}$ with $\gamma \in \Gamma$.

(iv) Applying (11) again we see that for any $\gamma \in \Gamma_0$ we have:

$$dt^{-\mathbf{1}_{\sigma(\gamma)}} = t^{\mathbf{1}_\gamma} dt^{-\mathbf{1}_\gamma} \, .$$

Given $d : \Gamma \to K$ the problem is to find a criterion so that the series defined via the strong Leibniz rule and strong linearity make sense, i.e. the family of terms is **summable** (the union of the supports is well-ordered and to any value in this union corresponds only finitely many members of the family). Using Ramsey theory type arguments we show:

Theorem: d extends to a series derivation on K iff both of the following conditions hold:

(C1) for any strictly increasing sequence $\gamma_n \subset \Gamma$ and any sequence $\tau_n \subset G$ s.t. τ_n in the support of $d(t_{\gamma_n})/t_{\gamma_n}$ for all n, τ_n cannot be decreasing.

(C2) for any strictly decreasing sequences $\gamma_n \subset \Gamma$ and $\tau_n \subset G$ s.t. τ_n in the support of $d(t_{\gamma_n})/t_{\gamma_n}$ for all n, there is N s.t.$v_G(\tau_{N+1} - \tau_N) > \gamma_{N+1}$.

Example of a well-defined derivation on K obtained in this way: set d on Γ by $d(t_\gamma)/t_\gamma := t^{-1}{}_{\sigma(\gamma)}$.

Application: Axs solution of Schanuels conjecture holds: if $s_i - s(0) \in K$ are rationally independent for $i = 1, \cdots n$ then their exponentials are algebraically independent.

PART III

Of particular interest to us is the analysis of certain equivalence relations on the surreal numbers. Conway introduced and studied the ω-map to give a complete system $\omega^{\mathbf{No}}$ (:= the image of **No** under this map) of representatives of the Archimedean additive equivalence relation. Exploiting the convexity of the subclass of positive elements of each equivalence class, Gonshor describes such a representative ω^a as the *unique* surreal of minimal length in a given class. By a simple modification of their arguments, we first describe a complete system $\omega^{\omega^{\mathbf{No}}}$ of representatives of the Archimedean multiplicative equivalence relation.

We then introduce and study what we call the κ-map to give a complete system $\kappa_{\mathbf{No}}$ (:= the image of $\mathbf{No}$ under this map) of representatives of the exponential equivalence relation. We observe that:

$$\epsilon_{\mathbf{No}} \subset \kappa_{\mathbf{No}} \subset \omega^{\omega^{\mathbf{No}}} \subset \omega^{\mathbf{No}} \subset \mathbf{No}.$$

Finally, we introduce the notion of exp-log transseries (ELT) fields, which unifies the notion of transseries and exp-log series. We conjectured that $\mathbf{No}$ is an ELT field.

Notation and Terminology:

• **No** is endowed with a partial ordering called the simplicity ordering : a is simpler than b, write $a <_s b$, iff a is a proper initial sign subsequence of b.

• We use Conway "cut" notation for surreals. For a pair (L, R) with $L < R$ of "left" and "right" subsets of **No** we denote by $\langle L|R\rangle := a \in \mathbf{No}$ the unique surreal of minimal length representing the cofinality class of the cut. We call it the cut between L and R.

• For any surreal number a, the representation $a = \langle L_a \mid R_a\rangle$ of a is called the canonical cut of a. We also denote the canonical cut by $a = \langle a^L \mid a^R\rangle$ where a^L and a^R are general elements of the canonical sets $L_a := \{b \in \mathbf{No}\ ;\ b < a, b <_s a\}$ and $R_a := \{b \in \mathbf{No}\ ;\ b > a, b <_s a\}$.

• (Conway Normal Form) The map Ω sending a to ω^a extends exponentiation in base ω of ordinals. For $a \in \mathbf{No}$, ω^a is the representative of minimal length of its Archimedean equivalence class.

Corollary For any $a \in \mathbf{No}$, ω^{ω^a} is the representative of minimal length in its equivalence class of comparability.

• (generalised epsilon numbers) $\epsilon(\mathbf{No})$ is the proper class of all the fixed points of the map Ω : $\forall a \in \mathbf{No},\ \omega^{\epsilon_a} = \epsilon_a$.

We shall now introduce a new class strictly between the class of epsilon numbers and that of representatives of the comparability classes.

The kappa map.

We study the exponential (logarithmic) equivalence relation for surreal numbers as we did for EL series, when we considered the exponential rank.

Theorem The recursive formula

$$\forall a \in \mathbf{No},\ \kappa(a) = \kappa_a := \langle \exp^n(0),\ \exp^n(\kappa_{a^L}) \mid \log^n(\kappa_{a^R}) \rangle$$

(where it is understood that $n \in \mathbb{N}$) defines a map

$$\begin{array}{rccc} \kappa: & \mathbf{No} & \to & \mathbf{No} \\ & a & \mapsto & \kappa(a) := \kappa_a \end{array}$$

with values in $\mathbf{No}_{>0}^{\gg 1}$ and such that:

(i) for any $a, b \in \mathbf{No}$, $a < b \Rightarrow \kappa_a \ll_{\exp} \kappa_b$;

(ii) there is a uniformity property for this formula (i.e. the recursive formula does not depend on the choice of the cut for a).