NSM2006 Nonstandard Methods Congress, Pisa May 25-31, 2006.

June 6, 2006

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[1] Partially supported by the Natural Sciences and Engineering Research Council of Canada.

κ-bounded Exponential-Logarithmic Power Series Fields.

Abstract:

• Since Wilkie's result **[9]** (which established that the elementary theory $\mathrm{T}_{\exp}$ of $(\mathbb{R}, \exp)$ is model complete and o-minimal), many o-minimal expansions of the reals have been investigated. The problem of constructing nonarchimedean models of $\mathrm{T}_{\exp}$ (and more generally, of an o-minimal expansion of the reals) gained much interest.

• In **[2]** it was shown that fields of generalized power series cannot admit an exponential function.

• Elaborationg on an idea of **[3]**, we construct in **[4]** fields of generalized power series with *support of bounded cardinality* which admit an exponential.

• **In this talk**, we present the construction given in **[4]**: We give a natural definition of an exponential, which makes these fields into models of the o-minimal expansion $T_{\mathrm{an,exp}}$:= the theory of the reals with restricted analytic functions and exponentiation.

• We present preliminary ideas on how to introduce derivation operators on these models. The aim is to present a new class of ordered differential fields, with many interesting properties.

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Notations and Preliminaries.

The natural valuation.

• Let G be a totally ordered abelian group. The archimedean equivalence relation on G is defined as follows. For $0 \neq x$, $0 \neq y \in G$:

$$x \stackrel{+}{\sim} y \text{ if } \exists n \in \mathbb{N} \text{ s.t. } n|x| \geq |y| \text{ and } n|y| \geq |x|$$

where $|x| := \max\{x, -x\}$. We set $x << y$ if for all $n \in \mathbb{N}$, $n|x| < |y|$. We denote by $[x]$ is the archimedean equivalence class of x. We totally order the set of archimedean classes as follows: $[y] < [x]$ if $x << y$.

• Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. Using the archimedean equivalence relation on the ordered abelian group $(K, +, 0, <)$, we can endow K with the **natural valuation** v:

for $x, y \in K, x, y \neq 0$ define $v(x) := [x]$ and $[x]+[y] := [xy]$.

Notation:

Value group: $v(K) := \{v(x) \mid x \in K, x \neq 0\}$.
Valuation ring:, $R_v := \{x \mid x \in K \text{ and } v(x) \geq 0\}$.
Valuation ideal: $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$.
Group of positive units:
$U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$.

Ordered Exponential Fields.

An ordered field K is an **exponential field** if there exists a map

$$\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$$

such that exp is an isomorphism of ordered groups. A map exp with these properties will be called an **exponential** on K. A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. WLOG, we require the exponentials (logarithms) to be v**-compatible**:

$$\exp(R_v) = U_v^{>0} \text{ or } \log(U_v^{>0}) = R_v > .$$

We are interested in exponentials (logarithms) satisfying the **growth axiom** scheme: **(GA)**:

$$\forall n \in \mathbb{N} : \ x > \log(x^n) = n\log(x) \text{ for all } x \in K^{>0} \setminus R_v .$$

Via the natural valuation v, this is equivalent to

$$v(x) < v(\log(x)) \text{ for all } x \in K^{>0} \setminus R_v . \tag{1}$$

A logarithm log is a **(GA)-logarithm** if it satisfies (1).

Hahn Groups and Fields.

• Let Γ be any totally ordered set and R any ordered abelian group. Then R^Γ is the set of all maps g from Γ to R such that the **support** $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ. Endowed with the lexicographic order and pointwise addition, R^Γ is an ordered abelian group, called the **Hahn group**.

• **Representation for the elements of Hahn groups:** Fix a strictly positive element $\mathbf{1} \in R$ (if R is a field, we take $\mathbf{1}$ to be the neutral element for multiplication). For every $\gamma \in \Gamma$, we will denote by $\mathbf{1}_\gamma$ the map which sends γ to $\mathbf{1}$ and every other element to 0 ($\mathbf{1}_\gamma$ is the characteristic function of the singleton $\{\gamma\}$.) For $g \in R^\Gamma$ write

$$g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$$

(where $g_\gamma := g(\gamma) \in R$).

• For $G \neq 0$ an ordered abelian group, k an archimedean ordered field, $k((G))$ is the (generalized) **power series field** with coefficients in k and exponents in G. As an ordered abelian group, this is just the Hahn group k^G. A series $s \in k((G))$ is written

$$s = \sum_{g\in G} s_g t^g$$

with $s_g \in k$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$.

• The natural valuation on $k((G))$ is $v(s) = \min \text{support}\, s$ for any series $s \in k((G))$. The value group is G and the residue field is k. The valuation ring $k((G^{\geq 0}))$ consists of the series with non-negative exponents, and the valuation ideal $k((G^{>0}))$ of the series with positive exponents. The **constant term** of a series s is the coefficient s_0. The units of $k((G^{\geq 0}))$ are the series in $k((G^{\geq 0}))$ with a non-zero constant term.

• **Additive Decomposition** Given $s \in k((G))$, we can truncate it at its constant term and write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus a complement in $(k((G)), +)$ to the valuation ring is the Hahn group $k^{G^{<0}}$. We call it the **canonical complement to the valuation ring** and denote it by $k((G^{<0}))$.

• **Multiplicative Decomposition** Given $s \in k((G))^{>0}$, we can factor out the monomial of smallest exponent $g \in G$ and write $s = t^g u$ with u a unit with a positive constant term. Thus a complement in $(k((G))^{>0}, \cdot)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g. We call it the **canonical complement to the positive units** and denote it by **Mon** $k((G))$.

κ–bounded Hahn Groups and Fields.

Fix a regular uncountable cardinal κ.

• The **κ-bounded Hahn group** $(R^\Gamma)_\kappa \subseteq R^\Gamma$ consists of all maps of which support has cardinality $< \kappa$.

• The **κ-bounded power series field** $k((G))_\kappa \subseteq k((G))$ consists of all series of which support has cardinality $< \kappa$. It is a valued subfield of $k((G))$. We denote by $k((G^{\geq 0}))_\kappa$ its valuation ring. Note that $k((G))_\kappa$ contains the monic monomials. We denote by $k((G^{<0}))_\kappa$ the complement to $k((G^{\geq 0}))_\kappa$.

• **Our first goal** is to define an exponential (logarithm) on $k((G))_\kappa$ (for appropriate choice of G). From the above discussion, we get:

Proposition 0.1 *Set $K = k((G))_\kappa$. Then $(K, +, 0, <)$ decomposes lexicographically as the sum:*

$$(K, +, 0, <) = k((G^{<0}))_\kappa \oplus k((G^{\geq 0}))_\kappa \ . \tag{2}$$

$(K^{>0}, \cdot, 1, <)$ decomposes lexicographically as the product:

$$(K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0} \tag{3}$$

Moreover, $\mathrm{Mon}\,(K)$ is order isomorphic to G through the isomorphism $t^g \mapsto -g$.

Proposition 0.1 allows us to achieve our goal in two main steps; by defining the logarithm on $\mathrm{Mon}\,(K)$ and on $U_v^{>0}$.

The Main Theorem

Theorem 0.2 *Let* Γ *be a chain,* $G = (\mathbb{R}^\Gamma)_\kappa$ *and* $K = R((G))_\kappa$*. Assume that*

$$l : \Gamma \to G^{<0}$$

is an embedding of chains. Then l *induces an embedding of ordered groups (a prelogarithm)*

$$\mathit{log} \; : (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$$

as follows: given $a \in K^{>0}$*, write* $a = t^g r(1+\varepsilon)$ *(with* $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$*,* $r \in \mathbb{R}^{>0}$*,* ε *infinitesimal), and set*

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i} \tag{4}$$

This prelogarithm satisfies

$$v(\log t^g) = l(\min \operatorname{support} g) \tag{5}$$

Moreover, log *is surjective (a logarithm) if and only if* l *is surjective, and* log *satisfies* **GA** *if and only if*

$$l(\min \operatorname{support} g) > g \qquad \text{for all } g \in G^{<0} \,. \tag{6}$$

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Prelogarithmic fields of power series.

Example 0.3 *Power Series fields endowed with a basic prelogaritm*: Let Γ be any chain, $G = (\mathbb{R}^\Gamma)_\kappa$ and $K = \mathbb{R}((G))_\kappa$. Then

$$\iota : \Gamma \to G^{<0} \text{ defined by } \gamma \mapsto -\mathbf{1}_\gamma$$

is an embedding of chains, and gives rise to prelogaritm on K. However, this prelogarithm is neither surjective nor does it satisfy **GA**.

• To get a prelogarithm satisfying **GA**, we choose $\sigma \in \mathrm{Aut}\,(\Gamma)$ with the property that

$$\sigma(\gamma) > \gamma \text{ for all } \gamma \in \Gamma \tag{7}$$

(We say that σ is an **increasing** automorphism). We set $l = \iota \circ \sigma$. Now

$$l : \Gamma \to G^{<0} \text{ defined by } \gamma \mapsto -\mathbf{1}_\sigma(\gamma)$$

is an embedding of chains satisfying (6), so gives rise to a prelogaritm on K satisfying **GA**.

We call $(K, \log)$ the **prelogarithmic field of κ-bounded power series over** (Γ, σ).

• To get a surjective prelogarithm, we have to modify Γ as in the next section.

The κ-th iterated lexicographic power of a chain.

Proposition 0.4 *Let $\Gamma \neq \emptyset$ be a given chain. There is a canonically constructed chain $\Gamma_\kappa \supseteq \Gamma$ together with an* **isomorphism** *of ordered chains*

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

where $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$. Moreover, every increasing $\sigma \in \mathrm{Aut}\,(\Gamma)$ extends canonically to an increasing $\sigma_\kappa \in \mathrm{Aut}\,(\Gamma_\kappa)$

We call the pair $(\Gamma_\kappa, \iota_\kappa)$ the κ-th **iterated lexicographic power** of Γ.

We are now ready to summarize the procedure of constructing the **Exponential-Logarithmic field of κ-bounded series over** (Γ, σ). Let Γ be given and σ an increasing automorphism.

- Construct Γ_κ, G_κ, ι_κ, and σ_κ.
- Set $K := \mathbb{R}((G_\kappa))_\kappa$ and $l := \iota_\kappa \circ \sigma_\kappa$. Note that l is surjective and satisfies (6).
- Denote by log the surjective **GA** logarithm induced on $K^{>0}$ by l and set $\exp = \log^{-1}$.
- $(K, \exp)$ is a model of $T_{\mathrm{an,exp}}$.

Growth Rates.

• Let Γ be a chain and $\sigma \in \mathrm{Aut}\,(\Gamma)$ an increasing automorphism. By induction, we define the **n-th iterate** of σ: $\sigma^1(\gamma) := \sigma(\gamma)$ and $\sigma^{n+1}(\gamma) := \sigma(\sigma^n(\gamma))$. Define an equivalence relation on Γ as follows: For $\gamma, \gamma' \in \Gamma$, set

$$\gamma \sim_\sigma \gamma' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \sigma^n(\gamma) \geq \gamma' \text{ and } \sigma^n(\gamma') \geq \gamma\ .$$

The equivalence classes $[\gamma]_\sigma$ of $\sim_\sigma$ are convex and closed under application of σ (they are the convex hulls of the orbits of σ). The order of Γ induces an order on $\Gamma/\sim_\sigma$. The order type of $\Gamma/\sim_\sigma$ is the **rank** of (Γ, σ).

Example 0.5 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$ (i.e. the lexicographically ordered Cartesian product $\mathbb{Z} \times \mathbb{Z}$) endowed with the automorphism $\sigma((x, y)) := (x, y + 1)$. The rank of σ is $\mathbb{Z}$. Now consider the increasing automorphism
$\tau((x, y)) := (x + 1, y)$. The rank of τ is 1.

• Let K be a real closed field and log a (**GA**)- logarithm on $K^{>0}$. Define an equivalence relation on $K^{>0} \setminus R_v$:

$$a \sim_{log} a' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \log_n(a) \leq (a') \text{ and } \log_n(a') \leq a$$

(where $\log_n$ is the n-th iterate of the log). The order type of the chain of equivalence classes is the **logarithmic rank** of $(K^{>0}, \log)$.

We can compute the logarithmic rank of the Exponential-Logarithmic field of κ-bounded series over (Γ, σ):

Theorem 0.6 *The logarithmic rank of* $(\mathbb{R}((G_\kappa))_\kappa^{>0}, \log)$ *is equal to the rank of* (Γ, σ).

This proof (as many other proofs) is based on the observation that every series is log-equivalent to a **fundamental monomial**, that is a monomial of the form

$$t^{-\mathbf{1}_\gamma} \text{ with } \gamma \in \Gamma \,.$$

Next one observes that

$$\text{for all } \gamma, \gamma' \in \Gamma : t^{-\mathbf{1}_\gamma} \sim_{log} t^{-\mathbf{1}_{\gamma'}} \text{ if and only if } \gamma \sim_\sigma \gamma' \,.$$

This in turn is based on the following useful formula for $\log_n(t^{-\mathbf{1}_\gamma})$: by induction,

$$\log{}_n(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma^n(\gamma)}} \,.$$

Remark 0.7 If Γ admits automorphisms of distinct rank, then $(\mathbb{R}((G_\kappa))$ admits logarithms of distinct logarithmic rank. We can also use this observation to introduce **transexponentials**, as illustrated in the next example.

Example 0.8 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$, $\sigma((x,y)) := (x, y+1)$, $(K, \log)$ the corresponding κ-bounded model. For the automorphism $\tau((x,y)) := (x+1, y)$, let L, respectively $T := L^{-1}$ be the corresponding induced logarithm and exponential on K.

Effect of σ, τ on the fundamental monomials: let $\gamma = (x,y) \in \Gamma$, then

$$\log(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma(\gamma)}} ,$$

Whereas

$$L(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\tau(\gamma)}} ,$$

We see that, for any fundamental monomial $X := t^{-\mathbf{1}_\gamma}$ and any $n \in \mathbb{N}$ we have:

$$L(X) < \log_n(X) .$$

Also, a simple computation (using the fact that σ and τ commute) shows that also, for all $n \in \mathbb{N}$:

$$T(X) > \exp_n(X) .$$

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In the next section, we see how the logarithm determines the derivation. We expect to obtain fields equipped with several distinct derivations.

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Introducing Operators.

Main Motivation: We want a "Kaplansky embedding Theorem" for ordered differential fields. The κ-bounded fields of power series are good candidates as "universal domains". But for this to make sense, we need first to endow them with a good differential structure.

Main project: Given (Γ, σ), introduce, if possible, derivation and composition operators on Exponential-Logarithmic field of κ-bounded series over (Γ, σ).

It seems to be enough to focus on the following

Main subproject: Given (Γ, σ), introduce, if possible, derivation and composition operators on the prelogarithmic field of κ-bounded series over (Γ, σ).

Indeed, in **[6]** a method is developed showing the following: given derivation and composition operators (satisfying some good properties) on a "field of transseries" $\mathbf{T}$, one can extend these operators to the "exponential closure" $\mathbf{T}^{\exp}$. It seems that this method may be adapted to our context: given derivation and composition operators (satisfying some good properties) on the prelogarithmic field of κ-bounded series over (Γ, σ), one can extend these operators to the Exponential-Logarithmic field of κ-bounded series over (Γ, σ).

Final Goal Find necessary and sufficient conditions on (Γ, σ) so that the corresponding prelogarithmic field of κ-bounded series (and the corresponding Exponential-Logarithmic field of κ-bounded series) admit a *surjective* derivation.

Derivations: We want to endow the prelogarithmic field of κ-bounded series over (Γ, σ) with a derivation D satisfying the following properties:

• D is strongly linear, that is

$$D \sum_g r_g t^g \;=\; \sum_g r_g D t^g \;. \tag{8}$$

• D satisfies Leibniz rule:

$$D(ab) = aD(b) + D(a)b \tag{9}$$

• D satisfies the rule for the logarithmic derivative for $a > 0$:

$$D \log a = Da/a \tag{10}$$

Reductions: The above rules direct us to perform a number of steps in trying to define derivatives:

(i) From (8) and (9), it is clear that we only need to determine Dt^g, for $g \in G^{<0}$.

(ii) From (10) determining Dt^g reduces to determining $D \log t^g$.

(iii) By definition of log, this in turn reduces to determining $D \log t^{-\mathbf{1}_\gamma}$, for a fundamental monomial $t^{-\mathbf{1}_\gamma}$ with $\gamma \in \Gamma$.

(iv) Applying (10) again we see that for any $\gamma \in \Gamma_0$ we have:

$$Dt^{-\mathbf{1}_{\sigma(\gamma)}} = t^{\mathbf{1}_\gamma} Dt^{-\mathbf{1}_\gamma} .$$

(v) Finally from (iv), we see that we only need to define $Dt^{-\mathbf{1}_{\gamma_0}}$ for a fixed representative $\gamma_0 \in \Gamma$ of an orbit of σ in Γ.

Example 0.9 Let $\Gamma = \mathbb{Z}$ endowed with the automorphism $\sigma(z) := z + 1$. For simplicity, let us choose $\gamma_0 = 0$ and set

$$T := t^{-\mathbf{1}_0} \text{ and } DT = 1 \,.$$

Then $t^{-\mathbf{1}_n} = \log_n T$ if $n > 0$, and $t^{-\mathbf{1}_n} = \exp_{-n} T$ if $n < 0$. Therefore, for $n > 0$

$$Dt^{-\mathbf{1}_n} = \prod_{k=0}^{n-1} t^{\mathbf{1}_k} \quad \text{and} \quad Dt^{-\mathbf{1}_{-n}} = \prod_{k=1}^{n} t^{-\mathbf{1}_{-k}} \,.$$

It is non-trivial to verify that these definitions induce a *well-defined* derivative!

Example 0.10 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$ endowed with the automorphism $\sigma((x, y)) := (x, y + 1)$. The rank of σ is $\mathbb{Z}$. For each orbit of σ_0 we fix a representative $z \in \mathbb{Z}$. We set $T_z := t^{-\mathbf{1}_z}$. Then $\{T_z \,;\, z \in \mathbb{Z}\}$ will represent infinitely many algebraically independent variables, which will determine an infinite family $\{\delta_z\}$ of commuting partial derivatives.

What about a derivation induced by the automorphism $\tau((x, y)) := (x+1, y)$ of rank one? This is more challenging. We have countably many distinct orbits but with a single common convex hull. This suggests defining " arbitrary iterates" $\log_\gamma T$ of the log, to capture the derivative of every fundamental monomial.

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