Test Sets for Positivity of Invariant Forms and Applications to Sums of Squares Representations

Salma Kuhlmann, University of Konstanz

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Plan of the talk[1]

[1] Dissertation of Ph.D. student Ms. Charu Goel

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1. Preliminaries and Hilbert's 17th Problem

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4. Version of Hilbert's 1888 Theorem for Even Symmetric Forms

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- But what if rational functions are not allowed in the sos representation and we want only sos of polynomials?

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- $(\mathcal{Q})$: For what pairs $(n, 2d)$ we have $\mathcal{P}_{n,2d} \subseteq \Sigma_{n,2d}$?

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Trivially, $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow f \in \mathcal{P}_{n+j,2d} \setminus \sum_{n+j,2d} \forall\ j \geq 0$. We claim: $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \sum_{n,\ 2d+2i}\ \forall\ i \geq 0$. Indeed, assume for a contradiction that $x_1^2 f(x_1, \ldots, x_n) = \sum_{j=1}^{k} h_j^2(x_1, \ldots, x_n)$. The L.H.S vanishes at $x_1 = 0$, so does the R.H.S. It follows that $h_j(x_1, \ldots, x_n)$ vanishes at $x_1 = 0$ and so $x_1 \mid h_j\ \forall\ j$, so $x_1^2 \mid h_j^2\ \forall\ j$. So, R.H.S is divisible by x_1^2.Dividing both sides by x_1^2 we get a sos representation of f, a contradiction.

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Conversely, Hilbert proved that $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, and observed:

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- **Proposition [BCR]:** Let R be a real closed field and p an irreducible polynomial in $R[x_1,\ldots,x_n]$. TFAE:
  1. $(p) = \mathcal{I}(Z(p))$, where $\mathcal{I}(A) = \left\{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \ \forall\ \underline{a} \in A\right\}$ is the ideal of vanishing polynomials on $A \subseteq R^n$ and $Z(p) = \left\{\underline{x} \in R^n \mid p(\underline{x}) = 0\right\}$ is the zero set of p.
  2. The sign of the polynomial p changes on R^n (i.e. $p(x)p(y) < 0$ for some $x, y \in R^n$).

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  - We will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$

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- **Corollary :** (i) For a symmetric real polynomial f of degree $2d$ in n variables $\exists\ \underline{x} \in \mathbb{R}^n$ s.t. $f(\underline{x}) = 0 \Leftrightarrow \exists\ \underline{x} \in \Lambda_{n,k}$ s.t. $f(\underline{x}) = 0$.

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so, $L_n(\underline{x}) = m(n-m)k(n-k)(r-s)^4 - [k(n-k)(r-s)^2]^2$

$= k(n-k)(r-s)^4[m(n-m) - k(n-k)]$

$= k(n-k)(r-s)^4[(m-k)(n-m-k)] \geq 0.$ $\square$

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  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n\geq 3}}_{\text{(C-L-R)}}, \underbrace{(3, 10), (4, 8)}_{\text{(Harris)}}$.

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  - construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n \geq 5}$
  - deduce that for $(n, 2d) = (n, 6)_{n \geq 3}$, $(n, 8)_{n \geq 4}, (3, 2d)_{d \geq 5}$, $(n, 2d)_{n \geq 4, d \geq 7}$, the answer to $\mathcal{Q}(S^e)$ is negative.

4.1. Degree jumping principle

- **Lemma 4.1 (G.):** If $2t = 4, 6$, and $n \geq 3$, then

$$h_t(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^{2t} - 10 \sum_{i \neq j} x_i^{2t-2} x_j^2$$

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- **Theorem 4.2 (G.)** [Degree jumping principle]:
  Let $n \geq 3$. If $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$, then
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  then the complete answer to $\mathcal{Q}(S^e)$ will be:

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  then the complete answer to $\mathcal{Q}(S^e)$ will be:

  $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$ if and only if $n = 2, d = 1, (n, 2d) = (n, 4)_{n \geq 3}, (3, 8)$.

- Psd not sos even symmetric $n-$ary octics for $n \geq 5$
  - **Theorem (G.):** The form

    $$B(x_1, \ldots, x_5) := L_5(x_1^2, \ldots, x_5^2) \in S\mathcal{P}^e_{5,8} \setminus S\Sigma^e_{5,8},$$

    (recall that $L_{2m+1} = m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2$ is a symmetric psd not sos $(2m+1)-$ary quartic form).

4.2.1. Psd not sos even symmetric $n-$ary octics for $n \geq 6$

- **Theorem (G.):** For $m \geq 3$,
  1. $M_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in S\mathcal{P}^e_{2m+1,8} \setminus S\Sigma^e_{2m+1,8}$, and
  2. $D_{2m} := C_{2m}(x_1^2, \ldots, x_{2m}^2) \in S\mathcal{P}^e_{2m,8} \setminus S\Sigma^e_{2m,8}$,

4.3. Version of Hilbert's 1888 Theorem for Even Symm forms

Theorem (G.):

1. $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ for $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}$, $(3, 8)$.
2. $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}$ and $(n, 2d)_{n\geq 4, d\geq 7}$.

i.e.

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	×	×	...
10	✓	×	?	?	?	?
12	✓	×	?	?	?	?
14	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

THANKS !