# The automorphism group of a valued Hahn field

Online Seminar talk by Salma Kuhlmann on Thursday October 26 2023

# I. Introduction

1. Motivation: Let $(K, v)$ a valued field, i.e. a field $K$ endowed with a valuation

$$v : K^\times \twoheadrightarrow G$$

where $G$ is a totally ord abelian group s.t. $v(ab) = v(a) + v(b)$

$$v(a+b) \geq \min\{v(a), v(b)\}$$

Want to study the group $v$-Aut $K$ of valuation preserving field and monomorphisms

$\sigma \in \mathrm{Aut}\,K$ is valuation preserving

if $\forall a, b \in K$

$v(a) = v(b) \Rightarrow v(\sigma(a)) = v(\sigma(b))$.

② We assume that $K$ admits a residue field and value group section

(residue field $R_v / I_v := Kv$

where $R_v$ is the val. ring,

i.e. $\{x \in K \mid v(x) \geq 0\}$

and $I_v := \{x \in K \mid v(x) > 0\}$)

$Kv \xhookrightarrow{i} K$ a field embedd.

$G \hookrightarrow (K^\times, \cdot)$ group embedding

Set Notation: $i(k) = k$.

 26.10.23

A Theorem of Kaplansky gives

$$(K, v) \xhookrightarrow{i} (k((G)), v_{min})$$

( $k((G))$ is the field of generalised power series or maximal Hahn field

$$k((G)) = \left\{ a = \sum_{g \in G} a_g t^g \;\middle|\; a_g \in k, \text{ support } a \text{ is a wellord in } G \right\}$$

$v_{min}(a) := \min \text{ support } a$ )

Moreover

$$k(G) \subseteq i(K) \subseteq k((G))$$

:= minimal Hahn field

$$:= ff(\underbrace{k[G]}_{:= \text{subring of } a \in k((G)) \text{ support } a \text{ is finite}})$$

(3) Therefore study the $v$-Aut $K$ where $K$ is a Hahn field, i.e

$$k(G) \subseteq K \subseteq k((G)).$$

Method: we will identify 2 properties, the 1st lifting property and the 2nd lifting property which help to describe $v$-Aut $K$.

a distinguished Hahn field is a Hahn field satisfies 1LP and 2LP.

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II. The 1 L P.

(1) In a paper we characterised the valuation preser. aut. of a valued field: if $\sigma \in v-\mathrm{Aut}\,K$ then

$$\sigma_G : v(a) \longmapsto v(\sigma(a))$$

(for $a \in K$), then $\sigma_G \in O\text{-}\mathrm{Aut}(G)$

$$\sigma_k : av \longmapsto \sigma(a)v$$

(for $a \in R_v$)

Then $\sigma_k \in \mathrm{Aut}(k)$.

(3) The $\Phi$-map.

Consider

$$\Phi : v-\mathrm{Aut}\,K \longrightarrow \mathrm{Aut}\,k \times O\text{-}\mathrm{Aut}\,G$$
$$\sigma \longmapsto (\sigma_k, \sigma_G)$$

$\Phi$ is a group homom.

$$\ker \Phi := \mathrm{Int\,Aut}\, K$$
$$= \{ \sigma \in v\text{-}\mathrm{Aut}\, K \mid$$
$$\forall a \in K : v(a) = v(\sigma(a))$$
$$\forall a \in R_v : c(a) = c(\sigma(a)) \}$$

where $c(a) :=$ constant term of $a$

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**Definition 1** $K$ has the ILP if $\Phi$ admits right section i.e a group hom.

$$\Psi : \mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut}\, G \longrightarrow v\text{-}\mathrm{Aut}\, K$$

such that $\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut}\, G}$

**Example 1** $k((G))$ admits the canonical lifting property

$$\Psi_c : \mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut}\, G \longrightarrow v\text{-}\mathrm{Aut}\, K$$

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$(\tau, \gamma) \longmapsto \sigma$ s.t

$$\sigma\left(\sum a_g t^g\right) \overset{!}{=} \sum \tau(a_g)\, t^{\gamma(g)} \quad (*)$$

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Definition 2: a Hahn field has the canonical lifting property if it is stable under $(*)$

Examples 2: The Rayner fields $k((\mathcal{F}))$, $\mathcal{F}$ family of wellord. subsets of $G$

Example 3: $k(G)$ has the canonical LP

(4) Definition 3: $\operatorname{im} \varphi :=$ Ext Aut $K$

(5) Fact (1) Int Aut $K \trianglelefteq$ v-Aut $K$

(2) Ext Aut $K \simeq$ Aut $k \times$ o-Aut $G$

Therefore

$$v\text{-}\mathrm{Aut}\, K \simeq \mathrm{Int\,Aut}\, K \rtimes \mathrm{Ext\,Aut}\, K$$

Therefore

1st Decomp. Theorem:

$$v\text{-}\mathrm{Aut}\, K \simeq \underbrace{\mathrm{Int\,Aut}\, K}_{} \rtimes \left(\underbrace{\mathrm{Aut}\, k \times \mathcal{O}\text{-}\mathrm{Aut}\, G}_{}\right)$$

!

Describe Int Aut K

Describing $\mathcal{O}$-Aut G (arXiv)

## III. The second lifting prop.

Consider the map

$$\mathcal{E} : \mathrm{Int\,Aut}\, K \longrightarrow \underbrace{\mathrm{Hom}(G, k^\times)}_{\substack{\text{Abelian group}\\ \text{under ptwise}\\ \text{multiplication}\\ \text{of characters}}}$$

$$\phi \longmapsto \underbrace{\left\{ \begin{array}{l} \chi_\sigma : \\ g \longmapsto e\left(\frac{\sigma(t^g)}{t^g}\right) \\ \forall g \in G \end{array} \right.}_{\chi_\sigma \in Hom(G, k^\times)}$$

The map $\mathcal{E}$ is a well defined group hom.

Consider

$\ker \mathcal{E} := 1 - \mathrm{Aut}\, K$

$= \{ \sigma \mid \forall a \in K : v(a) = v(\sigma(a))$ and the leading coeff. of $a$ = the " " of $\sigma(a) \}$

Definition 1 $K$ has the 2LP if $\mathcal{E}$ admits a right section $\rho$

i.e

$$\rho : Hom(G, k^\times) \longrightarrow Int\,Aut\,A$$

group hom.

and $\varepsilon \circ \rho = Id_{Hom(G,k^\times)}$

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Example 1: $k(G)$ 2LP

indeed $x \in Hom(G, k^\times)$

define

$$\phi_x\left(\sum a_g t^g\right) = \sum a_g\, x(g)\, t^g$$

**Definition 2** $K$ has the 2LP (**)

if invariant (**)

$$Int\,Aut\,K \cong \underbrace{1 - Aut\,K}\rtimes Hom(G, k^\times)$$

— $\mathcal{R}$. 26.10.23

$\sigma\text{-Aut}\, K \simeq$

$$\mathbb{1}\text{-Aut}\, K \rtimes \mathrm{Hom}(G, k^\times) \rtimes (\mathrm{Aut}\, k \times 0\text{-Aut}\, G)$$

$$= \!\!\!\! // \!\!\!\! =$$

End.

-9- 26.10.23