The Analogues of Hilbert's 1888 Theorem for Symmetric and Even Symmetric Forms

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Plan of the talk

1. Preliminaries on Hilbert's 17th Problem

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2. Hilbert's 1888 Theorem

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3. Hilbert's 1888 Theorem for Symmetric Forms

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4. Hilbert's 1888 Theorem for Even Symmetric Forms

1. Preliminaries on Hilbert's 17th Problem

- For $n \in \mathbb{N}$, a polynomial $p(x) \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called
  - **nonnegative or positive semidefinite (psd)** if $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$
  - a **sum of squares (sos)** if $p = \sum_i q_i^2$ for some $q_i \in \mathbb{R}[\underline{x}]$.
- Clearly every sos is psd from their representation.
- What about the converse?
- **Hilbert's 17th Problem:** Can we write every nonnegative polynomial p as a sum of squares of rational functions, i.e. $p = \sum_i \left(\frac{q_i}{r_i}\right)^2$ for some q_i, r_i (nonzero) $\in \mathbb{R}[\underline{x}]$?
- **Theorem of Artin and Schreier [1926]:** **YES** (over any real closed field).
- **Example:** The ternary sextic Motzkin form

$$M(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$$

is a sos of rational functions [e.g. $(x^2 + y^2 + z^2)^2 M(x, y, z)$ is sos].

1. Preliminaries Hilbert's 17th Problem

- But what if rational functions are not allowed in the sos representation and we want only sos of polynomials?
- When can a psd polynomial be written as a sos of polynomials?
- A polynomial $p \in \mathbb{R}[\underline{x}]$ of degree m is psd (resp. sos) iff its homogenization $p_h(x_0, x_1, \ldots, x_n) := x_0^m p(x_1/x_0, \ldots, x_n/x_0)$ is psd (resp. sos), so it is sufficient to consider this question for homogeneous polynomials, i.e. polynomials in which all terms have the same degree, also called **forms**. Let
- $\mathcal{F}_{n,m}$ be the vector space of all real forms in n variables and degree m, called **n-ary m-ics**, where $n, m \in \mathbb{N}$.
- $\mathcal{P}_{n,m} := \{f \in \mathcal{F}_{n,m} \mid f \text{ is psd }\}$, the cone of psd forms.
- $\Sigma_{n,m} := \{f \in \mathcal{F}_{n,m} \mid f \text{ is sos}\}$, the cone of sos forms.
- Since a psd form always has even degree, it is sufficient to study this question for even degree forms, so only consider $\mathcal{F}_{n,2d}$, $\mathcal{P}_{n,2d}$, $\Sigma_{n,2d}$.
- $(\mathcal{Q})$: For what pairs $(n, 2d)$ do we have $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$?

2. Hilbert's 1888 Theorem

- **Theorem (Hilbert, 1888):** $\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
- The arguments for the equality $\mathcal{P}_{n,2d} = \sum_{n,2d}$ for $n = 2$ and $d = 1$ were already known in the late 19th century (factorization theory of binary forms and diagonalization theorem of quadratic forms).
- For the equality $\mathcal{P}_{3,4} = \sum_{3,4}$, Hilbert showed that indeed every psd ternary quartic is a sum of at most three squares of quadratic forms.The idea of Hilbert's proof is to associate to any ternary quartic a curve and then use the classically well-developed theory of algebraic curves.
- Choi and Lam in 1977, gave an elementary proof of the equality $\mathcal{P}_{3,4} = \sum_{3,4}$, by exploiting extremal forms. They, however, did not show that only three quadratic forms suffice in such a sos representation.
- A modern simplified version of Hilbert's proof due to Cassels, was given by Rajwade in 1993, his proof shows that three squares suffice.

2. Hilbert's 1888 Theorem

For the only if direction, Hilbert established (abstractly) that

$$\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4} \text{ and } \Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$$

and observed:

- **Proposition 2.1[Reduction to Basic Cases]:**
  If $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$, then $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$

Proof.

Firstly, $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow f \in \mathcal{P}_{n+j,2d} \setminus \sum_{n+j,2d} \forall\, j \geq 0$. Secondly, we claim: $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \sum_{n,\ 2d+2i} \ \forall\ i \geq 0$. Indeed, assume for a contradiction that $x_1^2 f(x_1, \dots, x_n) = \sum_{j=1}^{k} h_j^2(x_1, \dots, x_n)$. The L.H.S vanishes at $x_1 = 0$, so does the R.H.S. It follows that $h_j(x_1, \dots, x_n)$ vanishes at $x_1 = 0$ and so $x_1 \mid h_j \ \forall\, j$, so $x_1^2 \mid h_j^2 \ \forall\, j$. So, R.H.S is divisible by x_1^2. Dividing both sides by x_1^2 we get a sos representation of f, a contradiction. Induction on i gives $x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \Sigma_{n,\ 2d+2i} \ \forall\ i \geq 1$. $\square$

2. Hilbert's 1888 Theorem

Examples of psd not sos ternary sextics and quaternary quartics:

- **Motzkin, 1967**
  $M(x,y,z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$
- **Robinson, 1969**
  $R(x,y,z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$,
  $W(x,y,z,w) := x^2(x-w)^2 + \big(y(y-w) - z(z-w)\big)^2 + 2yz(x+y-w)(x+z-w) \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$
- **Choi and Lam, 1976**
  $S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$
  $Q(x,y,z,w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$

3. Hilbert's 1888 Theorem for Symmetric forms

- A form $f \in \mathcal{F}_{n,2d}$ is called **symmetric** if $\forall\ \sigma \in S_n$: $f^\sigma(x_1,\ldots,x_n) := f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ is equal to $f(x_1,\ldots,x_n)$.
- $S\mathcal{P}_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is symmetric and psd}\}$
- $S\Sigma_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is symmetric and sos}\}$
- $\mathcal{Q}(S)$: For what pairs $(n, 2d)$ we have $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$?
- **Theorem (Choi and Lam, 1976):** $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.
- **Proposition 3.1 [Reduction to Basic Cases]**
  If $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4}$ for all $n \geq 4$ and $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$,then $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.
- **Proposition [BCR]:** Let R be a real closed field and p an irreducible polynomial in $R[x_1,\ldots,x_n]$. TFAE:
  1. $(p) = \mathcal{I}(Z(p))$, where $\mathcal{I}(A) = \left\{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \ \forall\ \underline{a} \in A\right\}$ is the ideal of vanishing polynomials on $A \subseteq R^n$ and $Z(p) = \left\{\underline{x} \in R^n \mid p(\underline{x}) = 0\right\}$ is the zero set of p.
  2. The sign of the polynomial p changes on R^n.

3. Hilbert's 1888 Theorem for Symmetric forms

- **Corollary 3.2:** Let $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$.
- Proof of Proposition 3.1 "Reduction to Basic Cases": If $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d}$, then $(x_1 + \ldots + x_n)^{2i} f \in S\mathcal{P}_{n,2d+2i} \setminus S\Sigma_{n,2d+2i}\ \forall\ i\ \geq 0$. $\square$
- Symmetric psd not sos ternary sextics and $n-$ary quartics for $n \geq 4$:
  - Robinson, 1969:
    $R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2 \in S\mathcal{P}_{3,6} \setminus S\Sigma_{3,6}$
  - Choi-Lam, 1976:
    $f_{4,4} := \sum^6 x^2y^2 + \sum^{12} x^2yz - 2xyzw \in S\mathcal{P}_{4,4} \setminus S\Sigma_{4,4}$. ["the construction of $f_{n,4} \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ (for $n \geq 4$) requires considerable effort, so we shall not go into the full details here. Suffice it to record the special form $f_{4,4}$."]
  - We will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$

3. Hilbert's 1888 Theorem for Symmetric forms

- **Timofte's Half Degree Principle for Symmetric Polynomials :** A symmetric real polynomial of degree $2d$ in n variables is nonnegative (> 0 respectively) on $\mathbb{R}^n \Leftrightarrow$ it is nonnegative (> 0 respectively) on the subset $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ number of distinct components in $\underline{x}$ is $\leq k$ $\}$, where $k := \max\{2, d\}$.
- A form $f \in \mathcal{F}_{n,2d}$ is called **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- **Timofte's Half Degree Principle for Even Symmetric Polynomials :** An even symmetric real polynomial of degree $2d \geq 4$ in n variables is nonnegative (> 0 respectively) on $\mathbb{R}^n \Leftrightarrow$ it is nonnegative (> 0 respectively) on the subset $\Omega_{n,d/2} := \{\underline{x} \in \mathbb{R}^n_+ \mid$ number of distinct nonzero components in $\underline{x}$ is $\leq d/2$ $\}$.
- **Corollary :** (i) For a symmetric real polynomial f of degree $2d$ in n variables $\exists\ \underline{x} \in \mathbb{R}^n$ s.t. $f(\underline{x}) = 0 \Leftrightarrow \exists\ \underline{x} \in \Lambda_{n,k}$ s.t. $f(\underline{x}) = 0$.
(ii) For an even symmetric real polynomial f of degree $2d$ in n variables $\exists\ \underline{x} \in \mathbb{R}^n$ s.t. $f(\underline{x}) = 0 \Leftrightarrow \exists\ \underline{x} \in \Omega_{n,d/2}$ s.t. $f(\underline{x}) = 0$.

3.1. Symmetric psd not sos $n-$ary quartics for $n \geq 5$

- Consider the following symmetric quartic in $n \geq 4$ variables,
  $$L_n(x_1, \ldots, x_n) := m(n-m)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2,$$
  where $m = [n/2]$.
- **Proposition 3.3:** L_n is psd for all n.
- **Theorem 3.4:** If $n \geq 5$ is odd, then L_n is not a sos.
- **Proposition 3.5:** L_n for even n is a sos.
  $$\Big[L_{2m}(\underline{x}) = \sum_{i<j}(x_i - x_j)^2\Big(-(x_1 + \ldots + x_{2m}) + m(x_i + x_j)\Big)^2\Big]$$
- For $m \geq 2$, consider the following symmetric quartic in $2m$ variables,
  $C_{2m}(x_1, \ldots, x_{2m}) := L_{2m+1}(x_1, \ldots, x_{2m}, 0)$.
- For $m \geq 2$, $C_{2m}(x_1, \ldots, x_{2m})$ is psd.
- **Theorem 3.6:** For $m \geq 2$, $C_{2m}(x_1, \ldots, x_{2m})$ is not a sos.

3.1. Symmetric psd not sos $n-$ary quartics for $n \geq 5$

- To prove: L_n is psd for all n.
  - $\Omega \subseteq \mathbb{R}^n$ is a **test set** for f if f is psd iff $f(\underline{x}) \geq 0$ for all $\underline{x} \in \Omega$.
  - **Theorem:** Let $n \geq 4$. A symmetric $n-$ary quartic f is psd iff $f(\underline{x}) \geq 0$ for every $\underline{x} \in \mathbb{R}^n$ with at most two distinct coordinates, i.e. $\Lambda_{n,2} := \{\underline{x} \in \mathbb{R}^n \mid x_i \in \{r, s\}; r \neq s, r, s \in \mathbb{R}\}$ is a test set for symmetric $n-$ary quartics.
- Proof: Enough to prove: $L_n \geq 0$ on $\Lambda_{n,2}$.

  Now for $\underline{x} \in \Lambda_{n,2} = \{(\underbrace{r, \ldots, r}_{k}, \underbrace{s, \ldots, s}_{n-k}) \mid r \neq s \in \mathbb{R};\ 0 \leq k \leq n\}$:

$$x_i - x_j = \begin{cases} \pm(r-s) \neq 0, & \text{for } k(n-k) \text{ terms,} \\ 0 & \text{, otherwise} \end{cases}$$

so, $L_n(\underline{x}) = m(n-m)k(n-k)(r-s)^4 - [k(n-k)(r-s)^2]^2$

$$= k(n-k)(r-s)^4[m(n-m) - k(n-k)]$$

$$= k(n-k)(r-s)^4[(m-k)(n-m-k)] \geq 0. \qquad \square$$

4. Version of Hilbert's 1888 Theorem for Even Symmetric forms

- $S\mathcal{P}^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and psd}\}$
- $S\Sigma^e_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is even symmetric and sos}\}$
- $\mathcal{Q}(S^e)$: For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?
- Known:
  - $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $\underbrace{n = 2, d = 1, (n, 2d) = (3, 4)}_{\text{(by Hilbert's Theorem)}}, \underbrace{(n, 4)_{n \geq 4}}_{\text{(C-L-R)}}, \underbrace{(3, 8)}_{\text{(Harris)}}$
  - $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = \underbrace{(n, 6)_{n \geq 3}}_{\text{(C-L-R)}}, \underbrace{(3, 10), (4, 8)}_{\text{(Harris)}}$.

4. Hilbert's 1888 Theorem for Even Symmetric forms

- To get a complete answer to $\mathcal{Q}(S^e)$ it is interesting to look at the following remaining cases:
  - $(3, 2d)$ for $d \geq 6$,
  - $(n, 8)$ for $n \geq 5$, and
  - $(n, 2d)$ for $n \geq 4, d \geq 5$.
- We will
  - give a "Reduction to Basic Cases" by finding an appropriate indefinite irreducible even symmetric form
  - construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n\geq 5}$
  - deduce that for $(n, 2d) = (n, 6)_{n\geq 3}$, $(n, 8)_{n\geq 4}, (3, 2d)_{d\geq 5}$, $(n, 2d)_{n\geq 4, d\geq 7}$, the answer to $\mathcal{Q}(S^e)$ is negative.

4.1. Degree jumping principle

- **Lemma 4.1:** If $2t = 4, 6$, and $n \geq 3$, then
$$h_t(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^{2t} - 10 \sum_{i \neq j} x_i^{2t-2} x_j^2$$
is an indefinite irreducible even symmetric $n-$ary form of degree $2t$.
- **Theorem 4.2** [Degree jumping principle]:
Let $n \geq 3$. If $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$, then
  1. for any integer $r \geq 2$, the form $f\, h_2^{2a}\, h_3^{2b} \in S\mathcal{P}^e_{n,2d+4r} \setminus S\Sigma^e_{n,2d+4r}$ where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$.
  2. $(x_1 \ldots x_n)^2 f \in S\mathcal{P}^e_{n,2d+2n} \setminus S\Sigma^e_{n,2d+2n}$.

4.2. Answer to $\mathcal{Q}(S^e)$: for what $(n, 2d)$ $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$?

- **Proposition (Reduction to Basic Cases:)** If we can find psd not sos even symmetric $n-$ary $2d-$ic forms for the following pairs:
  1. $(n, 2d) = (n, 8)$ for $n \geq 5$, and
  2. $(n, 2d)$ for $n \geq 4, d = 5, 6$.

  then the complete answer to $\mathcal{Q}(S^e)$ will be:

  $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$ if and only if $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$.

- Psd not sos even symmetric $n-$ary octics for $n \geq 5$
  - **Theorem:** The form

$$B(x_1, \dots, x_5) := L_5(x_1^2, \dots, x_5^2) \in S\mathcal{P}^e_{5,8} \setminus S\Sigma^e_{5,8},$$

(recall that $L_{2m+1} = m(m+1)\sum_{i<j}(x_i - x_j)^4 - \left(\sum_{i<j}(x_i - x_j)^2\right)^2$ is a symmetric psd not sos $(2m+1)-$ary quartic form).

4.2.1. Psd not sos even symmetric $n-$ary octics for $n \geq 6$

- **Theorem:** For $m \geq 3$,
  1. $M_{2m+1} := L_{2m+1}(x_1^2, \ldots, x_{2m+1}^2) \in S\mathcal{P}^e_{2m+1,8} \setminus S\Sigma^e_{2m+1,8}$, and
  2. $D_{2m} := C_{2m}(x_1^2, \ldots, x_{2m}^2) \in S\mathcal{P}^e_{2m,8} \setminus S\Sigma^e_{2m,8}$,

Set $M_r(x_1, \cdots, x_n) := x_1^r + \cdots x_n^r$. Use it to construct psd not sos even symmetric n-ary dedics and dodedics.

4.3. Hilbert's 1888 Theorem for Even Symmetric forms

Theorem:

1. $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ iff $n = 2, d = 1, (n, 2d) = (n, 4)_{n \geq 3},\ (3, 8)$.

i.e.

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	×	×	...
10	✓	×	×	×	×	×
12	✓	×	×	×	×	×
14	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

THANKS FOR YOUR INTEREST !