

# A NOTE ON $\aleph_\alpha$ -SATURATED O-MINIMAL EXPANSIONS OF REAL CLOSED FIELDS

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*Dedicated to Professor Yurii Ershov on his 75th birthday*

ABSTRACT. We give necessary and sufficient conditions for a polynomially bounded o-minimal expansion of a real closed field (in a language of arbitrary cardinality) to be  $\aleph_\alpha$ -saturated. The conditions are in terms of the value group, residue field, and pseudo-Cauchy sequences of the natural valuation on the real closed field. This is achieved by an analysis of types, leading to the trichotomy. Our characterization provides a construction method for saturated models, using fields of generalized power series.

## 1. INTRODUCTION

Let  $\mathcal{L}$  be a language containing  $<$ , an expansion of a real closed field  $\mathcal{M} = \langle M, +, \cdot, 0, 1, <, \dots \rangle$  is o-minimal if every subset of  $M$  which is definable with parameters in  $M$  is a finite union of intervals in  $M$ . For basic definitions and properties of o-minimal theories see [6].

In this paper we are interested in a valuation theoretic characterization of the  $\aleph_\alpha$ -saturated models of the o-minimal theory of  $\mathcal{M}$ . For the notion of Hausdorff's  $\eta_\alpha$ -sets, see [7]. Denote by  $|A|$  the cardinality of  $A$ , and by  $|\mathcal{L}| := \ell$  that of the language. Let  $\text{dcl}(A)$  denote the definable closure of  $A \subset M$ . We have:  $|\text{dcl}(A)| = \max\{|A|, \ell\}$ . O-minimality implies that a formula  $\varphi(x, \bar{a})$  in one variable  $x$  and parameters  $\bar{a}$  is equivalent to a disjunction of intervals with endpoints in  $\text{dcl}(\bar{a})$ , thus realizing a type in one variable  $x$  and parameters in  $A$  amounts to realizing a cut in the  $\text{dcl}(A)$ . Let  $|A| < \aleph_\alpha$ . In the case when  $\aleph_\alpha > \ell$ , then also  $|\text{dcl}(A)| < \aleph_\alpha$ . Therefore in that case  $\mathcal{M}$  is  $\aleph_\alpha$ -saturated if and only if it is  $\aleph_\alpha$ -saturated as a linear order, that is an  $\eta_\alpha$ -set. So our result gives also a characterization of  $\eta_\alpha$ -models (for  $\alpha$  with  $\aleph_\alpha > \ell$ ), extending [1].

We assume familiarity with the notions of independence and dimension in o-minimal structures. We note that  $\dim(\text{dcl}(A)) \leq |A|$ . For

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notions and results concerning power bounded o-minimal expansions needed below see [6]. We shall need the following fact. An o-minimal theory admits (up to isomorphism) a unique prime model  $\mathcal{P}$  with underlying set  $\text{dcl}(\emptyset) = P$ . For example, if  $\mathcal{M}$  is a divisible abelian group then  $\text{dcl}(A)$  in the  $\mathbb{Q}$ -vector space generated by  $A$ , the prime model is  $\mathbb{Q}$ , and the dimension is the  $\mathbb{Q}$ -linear dimension. If  $\mathcal{M}$  is a real closed field then  $\text{dcl}(A)$ , the relative algebraic closure of the field  $\mathbb{Q}(A)$  in  $M$ , the prime model is  $\mathbb{Q}^{rc}$ , the field of real algebraic numbers, and the dimension is the (absolute) transcendence degree.

Our main result in the polynomially bounded case, that is, power bounded with archimedean field of exponents, is Theorem 3.2 (see Remark 3.4 for the power bounded case).

Note that condition (1) in Theorem 3.2 can be replaced by the equivalent valuation theoretic characterization of  $\aleph_\alpha$ -saturated ordered  $Q$ -vector spaces, which we provide in Theorem 2.2. Theorem 3.2 generalizes [3, Theorem 6.2] which treated the special case when  $\mathcal{M}$  is just a real closed field.

## 2. THE CASE OF ORDERED $Q$ -VECTOR SPACES

Throughout this section we assume  $Q$  to be an Archimedean ordered field, so  $Q$  is a subfield of  $\mathbb{R}$ . We recall some general definitions about ordered  $Q$ -vector spaces, see [5] for more details. Let  $G$  be an ordered  $Q$ -vector space, for any  $x \in G$  let  $|x| = \max\{x, -x\}$ . For non-zero  $x, y \in G$  we say that  $x$  is  $Q$ -archimedean equivalent to  $y$  if there exists  $q \in Q$  such that  $q|x| \geq |y|$  and  $q|y| \geq |x|$ . We denote this relation by  $\sim_Q$ . We write  $x \ll_Q y$  if  $q|x| < |y|$  for all  $q \in Q$ . Clearly,  $\sim_Q$  is an equivalence relation. Let  $\Gamma := \{[x] : x \in G, x \neq 0\}$  the set of equivalence classes. We define a linear order  $<$  on  $\Gamma$  as follows,  $[y] < [x]$  if  $x \ll_Q y$  (notice the reversed order). The valuation associated to  $G$  as an ordered  $Q$ -vector space is the map  $v_Q: G \rightarrow \Gamma \cup \{\infty\}$  defined by  $v_Q(0) = \infty$  and  $v_Q(x) = [x]$  if  $x \neq 0$ . It satisfies the following axioms:  $v_Q(x) = \infty$  if and only if  $x = 0$ ;  $v_Q(qx) = v_Q(x)$  for all  $q \neq 0$ ; and  $v_Q(x-y) \geq \min\{v_Q(x), v_Q(y)\}$  (and equality holds for distinct values). We call  $\Gamma$  the value set of  $G$ .

We shall need the following valuation inequality for ordered  $Q$ -vector spaces. For the discussion below, let us fix a set of representatives  $0 \neq g_\alpha \in G$  such that  $\gamma_\alpha = v_Q(g_\alpha)$ .

**Lemma 2.1.** *The cardinality of the value set  $\Gamma$  is less or equal than the dimension of  $G$  over  $Q$ , i.e.  $|\Gamma| \leq \dim_Q(G)$ .*

*Proof.* Let  $\Gamma = \{\gamma_\alpha : \alpha < \kappa\}$  and  $|\Gamma| = \kappa$ , for a cardinal  $\kappa$ . We claim that  $\{g_\alpha : \alpha < \kappa\}$  are  $Q$ -independent. If not, there are  $g_1, \dots, g_n$  and  $q_1, \dots, q_n \in Q^\times$  such that  $\sum_{i=1}^n q_i g_i = 0$ . Now since the values are pairwise distinct, we have  $\infty = v_Q(\sum_{i=1}^n q_i g_i) = \min\{v_Q(g_i) : i = 1, \dots, n\}$ , a contradiction.  $\square$

For every  $\gamma \in \Gamma$ , fix  $A_\gamma$  a maximal archimedean  $Q$ -subspace of  $G$  containing  $g_\gamma$ , the archimedean component associated to  $\gamma$ . We recall that, for a limit ordinal  $\lambda$ , a sequence  $(a_\rho)_{\rho < \lambda}$  is pseudo Cauchy if for every  $\rho < \sigma < \tau < \lambda$  we have  $v_Q(a_\sigma - a_\rho) < v_Q(a_\tau - a_\sigma)$ , and  $a \in G$  is a pseudo limit if for all  $\rho < \lambda$ ,  $v_Q(a - a_\rho) = v_Q(a_{\rho+1} - a_\rho)$ .

We now give a characterization of  $\aleph_\alpha$ -saturation for ordered  $Q$ -vector spaces in the language  $\mathcal{L}_Q$  of ordered groups  $\mathcal{L}_{OG}$  expanded with constants for the elements of  $Q$ , i.e.  $\mathcal{L}_Q = \mathcal{L}_{OG} \cup \{c_q : q \in Q\}$ . This is a generalization of the characterization for divisible ordered abelian groups, see [4].

**Theorem 2.2.** *Let  $G$  be an ordered  $Q$ -vector space. Then  $G$  is  $\aleph_\alpha$ -saturated in the language  $\mathcal{L}_Q$  if and only*

- (1) *its value set  $\Gamma$  is an  $\eta_\alpha$ -set*
- (2) *all its Archimedean components are isomorphic to  $\mathbb{R}$*
- (3) *every pseudo Cauchy sequence in a  $Q$ -subspace of  $G$  of dimension  $< \aleph_\alpha$  has a pseudo limit in  $G$ .*

We note that the characterization provides a method for constructing  $\aleph_\alpha$ -saturated ordered  $Q$ -vector spaces, using Hahn group constructions (see [5]). Let  $\Gamma$  be any  $\eta_\alpha$ -set, and let  $G = \Pi_\Gamma \mathbb{R}$  the Hahn product. Then  $G$  is  $\aleph_\alpha$ -saturated.

### 3. THE CASE OF POWER BOUNDED O-MINIMAL EXPANSIONS

If  $R$  is an ordered field then its natural valuation  $v$  has valuation ring the convex hull of  $\mathbb{Q}$  in  $R$ . The residue field  $k$  is archimedean, i.e. (isomorphic to) a subfield of  $\mathbb{R}$ . We denote the value group  $v(R)$  by  $G$ . Note that  $G$  is divisible and its rational rank is the linear dimension as a  $Q$ -vector space. See [5] for details. A characterization of  $\aleph_\alpha$ -saturated real closed fields was obtained in [3, Theorem 6.2]. We note that in the proof of [3, Theorem 6.2], the dimension inequality (the rational rank of the value group is bounded by the absolute transcendence degree; see [2]) is crucially used. In this Section, we prove a generalization of this result to o-minimal expansions of a real closed field  $\mathcal{M} = (M, +, \cdot, 0, 1, <, \dots)$ . Recall that the expansion  $\mathcal{M}$  is power bounded if for each definable function  $f : \mathcal{M} \rightarrow \mathcal{M}$  there is  $\lambda \in M$  such that  $|f(x)| \leq x^\lambda$  for all sufficiently large  $x > 0$  in  $M$ . We denote the field of exponents (of all definable power functions) by  $Q$ .

**Remark 3.1.** A power bounded  $\mathcal{M}$  is polynomially bounded if and only if  $Q$  is archimedean. We shall assume that the prime model (of the theory of  $\mathcal{M}$ ) is archimedean. In that case, the valuation ring of the natural valuation is  $T$ -convex and the value group  $G$  is an ordered  $Q$ -vector space. The following analogue of the dimension inequality (the valuation inequality) holds:  $\dim_Q G \leq \dim(\mathcal{M})$ . In particular,  $\dim_Q(v(\text{dcl}(A))) \leq \dim(\text{dcl}(A)) \leq |A|$ . See [6] for details.

**Theorem 3.2.** *Let  $\mathcal{M} = \langle M, <, +, \cdot, \dots \rangle$  be a polynomially bounded  $o$ -minimal expansion of a real closed field,  $v$  its natural valuation,  $G$  its value group,  $k$  its residue field. We assume that its prime model  $\mathcal{P}$  is archimedean. Then  $\mathcal{M}$  is  $\aleph_\alpha$ -saturated if and only if*

- (1)  $G$  is  $\aleph_\alpha$ -saturated as an ordered  $Q$ -vector space
- (2)  $k \cong \mathbb{R}$
- (3) for every substructure  $\mathcal{M}'$  with  $\dim(\mathcal{M}'/\mathcal{P}) < \aleph_\alpha$ , every pseudo Cauchy sequence in  $M'$  has a pseudo limit in  $M$ .

*Proof.* The proof follows the lines of the proof of [3, Theorem 6.2], and we shall only point out the main adjustments which concern the cardinality of the expanded language, and bounding the dimension of definable closures. We begin by observing that the proof of necessity is analogous to that of [3, pp.88-89]; one needs to take  $Q$ -linear spans instead of  $\mathbb{Q}$ -linear spans. Now assume conditions (1), (2) and (3) and we show that  $\mathcal{M}$  is  $\aleph_\alpha$ -saturated. Let  $q$  be a complete 1-type over  $\mathcal{M}$  with parameters in  $A \subset M$ , with  $|A| < \aleph_\alpha$ . Let  $\mathcal{M}''$  be an elementary extension of  $\mathcal{M}$  in which  $q(x)$  is realized, and  $x_0 \in M''$  such that  $\mathcal{M}'' \models q(x_0)$ . As noted earlier to realize  $q$  in  $\mathcal{M}$  it is necessary and sufficient to realize the cut that  $x_0$  makes in  $\mathcal{M}' := \text{dcl}(A) \subseteq \mathcal{M}$

$$q'(x) := \{b \leq x; b \in M', q \vdash b \leq x\} \cup \{x \leq c; c \in M', q \vdash x \leq c\}.$$

As we will see in realizing the cut  $q'$  instead of the type  $q$  some case distinction will be needed according to whether  $\aleph_\alpha > \ell$  (so  $|\text{dcl}(A)| < \aleph_\alpha$ ) or  $\aleph_\alpha \leq \ell$  (so  $|\text{dcl}(A)| \geq \aleph_\alpha$ ). As in the proof of [3, Theorem 6.2] we set  $B := \{b \in M'; q \vdash b < x\}$  and  $C := \{c \in M'; q \vdash x < c\}$ . Consider  $\Delta = \{v(d - x_0) \mid d \in M'\}$  and the three possible cases:

- (a) *Immediate transcendental case:*  $\Delta$  has no largest element.
- (b) *Value transcendental case:*  $\Delta$  has a largest element  $\gamma \notin v(M')$ .
- (c) *Residue transcendental case:*  $\Delta$  has a largest element  $\gamma \in v(M')$ .

We deal with case (a) as in [3, p. 86], taking into account that we get a pseudo-Cauchy sequence in  $M'$  and that  $\dim(\mathcal{M}/\mathcal{P}) \leq |A| < \aleph_\alpha$ . In case (b) we need to deal with the type

$$t(y) = \{v(c - d_0) < y; c \in C\} \cup \{y < v(b - d_0); b \in B, b > d_0\}$$

over  $G$  with parameters in  $v(M')$  as in [3, p. 87]. Set  $G' = v(M')$ . If  $\aleph_\alpha > \ell$  then  $|G'| < \aleph_\alpha$  and by hypothesis (1) we can realize  $t(y)$  in  $G$ . The delicate case is when  $\aleph_\alpha \leq \ell$ . By Remark 3.1,  $\dim_Q(v(G')) \leq \dim(\mathcal{M}'/\mathcal{P}) \leq |A| < \aleph_\alpha$ . Fixing a  $Q$ -basis of  $G'$  of cardinality  $< \aleph_\alpha$ , we can rewrite  $t(y)$  as a type  $t'(y)$  with parameters now from the basis. Since  $G$  is  $\aleph_\alpha$ -saturated, we can realize  $t'(y)$  in  $G$ . The remaining of the argument, as well as the argument for case (c) are as in [3, pp. 87-88].  $\square$

We note that the characterization provides a construction method for  $\aleph_\alpha$ -saturated polynomially bounded o-minimal expansions of real closed fields, using (the maximally valued) fields of power series.

**Example 3.3.** Consider the uncountable language  $\mathcal{L}_{an}$  of ordered fields expanded with symbols for restricted analytic functions. Construct the power series field  $\mathbb{R}((G))$  (see [5]) where  $G$  is the  $\ell$ -saturated group constructed at the end of previous section. The field  $\mathbb{R}((G))$  can be endowed with a polynomially bounded o-minimal  $\mathcal{L}_{an}$ -structure (see [6]). By Theorem 3.2 it is  $\ell$ -saturated.

**Remark 3.4.** (1) Assume that  $\mathcal{M}$  is polynomially bounded but the prime model is non archimedean. Consider the coarsening  $w$  of  $v$  whose valuation ring is the convex hull of the prime model in  $M$ . Working with this valuation, one gets an analogue of Theorem 3.2.

(2) Assume now that  $\mathcal{M}$  is power bounded, but not polynomially bounded, i.e. the field of exponents  $Q$  is non-archimedean. In this case, working with the induced valuation  $w_Q$  on the ordered  $Q$ -vector space  $w_Q(K)$ , one gets analogues of Theorems 2.2 and 3.2.

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