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Liste des exposés.

14.11.95 - **Luc Bélair** (Univ. du Québec, Montréal, Canada, et Univ. Paris 7).
Types selon Delon.

21.11.95 - **Françoise Delon** (Univ. Paris 7).
Ax-Kochen-Ershov en caractéristique positive.

28.11.95 - **Deirdre Haskell** (Holy Cross College, Etats-Unis, et Univ. Paris 7).
Une variante de l'o-minimalité pour les p-adiques.

09.01.96 - **Patrick Simonetta** (Univ. Paris 7).
Un principe d'Ax-Kochen-Ershov pour des groupes valués.

16.01.96 - **Françoise Delon** (Univ. Paris 7).
Ax-Kochen-Ershov en caractéristique positive (suite).

23.01.96 -**Vincent Astier** et **Max Dickmann** (Univ. Paris 7).
Résultats modèle-théoriques en théorie des formes quadratiques.

30.01.96 - **Aldo Ursini** (Univ. de Sienne, Italie)
Quantales and Girales .

13.02.96 - **Francisco Miraglia** (Univ. Sào Paulo, Brésil, et Univ. Paris 7).
A measure theoretic Erdös-Rado theorem.

20.02.96 - **Eberhard Becker** (Univ. Dortmund, Allemagne).
The higher "Stufe" of finite fields and rings.

27.02.96 - **Francisco Miraglia** (Univ. Sào Paulo, Brésil, et Univ. Paris 7).
Relations between Milnor's K-theory for quadratic forms and special groups.

19.03.96 - **Jorge Guier** (Univ. de Costa-Rica et Univ. Paris 7).
Produits booléens de corps réels clos et d'anneaux réels clos.

26.03.96 - **Yerzhan Baisalov** (Univ. d'Almaty, Kazakhstan, et Univ. Lyon 1).
Paires de structures o-minimales.

28.05.96 - **Charles Delzell** (Baton-Rouge, Louisianne, Etats-Unis).
A completely normal spectral space that is not a real spectrum.

04.06.96 - **Maria Jesus de la Puente** (Univ. Complutense, Madrid, Espagne).
A complex version of the Baer-Krull theorems.

11.06.96 - **Salma Kuhlmann** (Univ. Heidelberg, Allemagne).
Le rang exponentiel.

24.06.96 - **Niels Schwartz** (Univ. Passau, Allemagne).
Real-closed rings. Examples and applications.

25.06.96 - **Niels Schwartz** (Univ. Passau, Allemagne).
Sums of squares in a formally real field.

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Liste des contributions.

Luc Bélair et Michel Bousquet (Univ. du Québec, Montréal, Canada).
Types dans les corps valués.

Françoise Delon (Univ. Paris 7).
Corps avec ou sans la propriété d'indépendance.

Deirdre Haskell (Holy Cross College, Massachussets, E.U.).
Quelques variantes de l'o-minimalité pour des corps valués.

Françoise Delon et Patrick Simonetta (Univ. Paris 7).
Un principe d'Ax-Kochen-Ershov pour des structures intermédiaires entre groupes et corps valués. (Abstract)

Vincent Astier (Univ. Paris 7).
Quelques propriétés modèle-théoriques des groupes spéciaux réduits.

Aldo Ursini (Univ. de Sienne, Italie).
Quantales and Girales. (Abstract)

Max Dickmann (Univ. Paris 7) **et Francisco Miraglia** (Univ. Sào Paulo, Brésil).
Algebraic K-theory of fields and special groups.

Jorge Guier (Univ. de Costa-Rica et Univ. Paris 7).
Produits booléens d'anneaux réels clos.

Yerzhan Baisalov (Académie des Sciences, Almaty, Kazakhstan) **et Bruno Poizat** (Univ. Lyon 1).
Paires de structures o-minimales.

Maria Jesus de la Puente (Univ. Complutense, Madrid, Espagne).
The complex spectrum of a ring.

Maria Jesus de la Puente (Univ. Complutense, Madrid, Espagne).
A complex version of the Baer-Krull theorems.

Franz-Viktor Kuhlmann, Salma Kuhlmann (Univ. Heidelberg, Allemagne, et Fields Institute, Toronto, Canada) **et Saharon Shelah** (Hebrew Univ., Jérusalem, Israël).
Exponentiation in power series fields.

Franz-Viktor Kuhlmann et Salma Kuhlmann (Fields Institute, Toronto, Canada).
Residue fields of arbitrary convex valuations on restricted analytic fields with exponentiation, I.

Franz-Viktor Kuhlmann et Salma Kuhlmann (Fields Institute, Toronto, Canada).
Explicit construction of exponential-logarithmic power series.

Franz-Viktor Kuhlmann (Fields Institute, Toronto, Canada).
Elementary properties of power series fields over finite fields.

Niels Schwartz (Univ. Passau, Allemagne).
Real-closed rings. Examples and applications. (Abstract)

Niels Schwartz (Univ. Passau, Allemagne).
The sums of squares in a formally real field. (Abstract)

Niels Schwartz (Univ. Passau, Allemagne).
Simple complete ideals in polynomial rings with two variables: monomial ideals.

Explicit construction of exponential-logarithmic power series *

— preliminary version —

Franz-Viktor and Salma Kuhlmann

12. 1. 1997

In this note, we give an explicit construction of nonarchimedean models of real exponentiation which are contained in power series fields. In contrast to the construction given by van den Dries, Macintyre and Marker in [D–M–M2], our construction uses only one limit process.

Recent developments in the model theory of exponential fields have shown that in certain respects, the logarithm plays a more basic role than the exponential. And in fact, every power series field $\mathbb{R}((G))$ carries a non-surjective logarithm (which cannot be surjective, as is shown in [K–K–S]). In [D–M–M2], a first limit process is employed to obtain a field with non-surjective exponential, and then a second (inverse) limit process renders the exponential surjective. Reversing the approach, we start with a non-surjective logarithm on a power series field and get it surjective by taking the union over an ascending chain of power series fields.

Apart from being simpler, our approach facilitates computations in the constructed models. Moreover, it exhibits the relation between order automorphisms of the value groups and the growth rates of the constructed exponentials. In particular, our construction can be used to provide models of arbitrary exponential rank, in the sense of [K–K1]. It also allows to obtain on one and the same real closed field K several exponentials of distinct exponential ranks, all of them making K into a model of real exponentiation. This contrasts the fact that the order and hence also the rank of a real closed field is uniquely determined.

A special case of our construction provides a model with similar properties as the model $\mathbb{R}((t))^{LE}$ constructed in [D–M–M2]. We denote this model by $\mathbb{R}((t))^{EL}$. (We do not know whether both models are isomorphic, but this does not seem unlikely.) We show how to obtain truncation closed embeddings in $\mathbb{R}((t))^{EL}$ of the Hardy fields considered in [D–M–M2]. We also show how to endow $\mathbb{R}((t))^{EL}$ with derivations.

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1 Construction of a basic logarithm

We will first construct the exponential-logarithmic power series together with a "basic logarithm" which is an isomorphism from the positive multiplicative group onto the additive group. Its inverse will satisfy all necessary axioms in order to obtain a model of real exponentiation, except for the growth axiom which we will cite in the next section. We will then modify the basic logarithm in order to obtain also the right growth.

Let Γ be any totally ordered set. Then $\mathbb{R}^\Gamma$ will denote the Hahn product with index set Γ and components $\mathbb{R}$, that is, all maps from Γ to $\mathbb{R}$ with well-ordered support. $\mathbb{R}^\Gamma$ is an ordered abelian group. For every $\gamma \in \Gamma$, we will denote by 1_γ the map which sends γ to 1 and every other element to 0. (1_α is the characteristic function of the singleton $\{\alpha\}$.) Note that the map $\Gamma \ni \gamma \mapsto -1_\gamma \in \mathbb{R}^\Gamma$ is an order preserving embedding of Γ in $(\mathbb{R}^\Gamma)^{<0} = \{g \in \mathbb{R}^\Gamma \mid g < 0\}$.

For $g \in \mathbb{R}^\Gamma$, we have that $r_\gamma := g(\gamma) \in \mathbb{R}$ for every $\gamma \in \Gamma$. Instead of viewing g as a map, let us work with the more suggestive expression $g = \sum_{\gamma\in\Gamma} r_\gamma 1_\gamma$; although this sum may be infinite, it has a canonical interpretation in the Hahn product $\mathbb{R}^\Gamma$.

For G an ordered abelian group, $\mathbb{R}((G))$ will denote the (generalized) power series field with coefficients in $\mathbb{R}$ and exponents in G. As an ordered abelian group, this is just $\mathbb{R}^G$. When we work in $\mathbb{R}((G))$, we will write t^g instead of 1_g. Hence, every element of $\mathbb{R}((G))$ can be written in the form $\sum_{g\in G} r_g t^g$ with $r_g \in G$ and well-ordered support $\{g \in G \mid r_g \neq 0\}$. (In [D–M–M2], the notation $\mathbb{R}((t^G))$ is used instead of $\mathbb{R}((G))$.)

In [D–M–M1], a canonical way is described of how to make $\mathbb{R}((G))$ into a model of the theory of the reals with restricted analytic functions (via defining the functions on $\mathbb{R}[[G]]$ by using their Taylor expansions.) In this way, one in particular obtains the function log sending the 1-units (the elements of the form $1 + \varepsilon$ where ε is an infinitesimal, i.e., $v\varepsilon > 0$) onto the additive subgroup of all infinitesimals. For positive units $u \in \mathbb{R}[[G]]$, we write $a = (1+\varepsilon)r$ with $v\varepsilon > 0$ and $0 < r \in \mathbb{R}$. Then log is extended by setting $\log u = \log(1+\varepsilon) + \log r$, where $\log r$ is the natural logarithm of r in $\mathbb{R}$. We obtain a monomorphism log from the units of $\mathbb{R}[[G]]$ onto $\mathbb{R}[[G]]$. The restriction of $\log^{-1}$ to $[-1, 1]$ coincides with the restricted analytic function on $\mathbb{R}((G))$ corresponding to the natural exp of the reals.

Let $G = \mathbb{R}^\Gamma$. Then $\mathbb{R}((G))$ carries a basic non-surjective logarithm, which we denote by $\log_0$ and define as follows. Take $a \in \mathbb{R}((G))$ positive and write $a = ut^g$ with $u \in \mathbb{R}[[G]]$ a unit and $g \in G$. Then for any homomorphism log from the positive multiplicative to the additive group, we must have that

$$\log a = \log(ut^g) = \log u + \log t^g \ .$$

For every logarithm log that we will introduce, we define $\log u$ as discussed above. It remains to give appropriate definitions for $\log t^g$. We will do this first for our basic logarithm $\log_0$. We choose it such that it leaves the elements t^{-1_γ} fixed:

$$\log_0 t^{-1_\gamma} = t^{-1_\gamma} \ . \tag{1}$$

For every appropriate logarithm log, we want to have that $\log t^{rg} = r \log t^g$ for every $r \in \mathbb{R}$ and $g \in G$, and that log is sort of "ultrametrically continuous". This leads to the following definition:

$$\text{if } g = \sum_{\gamma\in\Gamma} r_\gamma 1_\gamma \ , \ \text{then} \quad \log_0 t^g \ := \ \sum_{\gamma\in\Gamma} -r_\gamma t^{-1_\gamma} \ .$$

In a straightforward way it is checked that

$$ut^g \ \mapsto \ \log u + \log_0 t^g$$

is an order preserving monomorphism from the positive multiplicative group to the additive group. However, it is obviously not surjective: if $g \in G^{<0}$ is not of the form $g = -1_\gamma$, then t^g is not in the image of $\log_0$. In other words, for $g \in G^{<0}$, t^g is in the image of $\log_0$ if and only if g is in the image of the embedding of Γ in $G^{<0}$ given by $\gamma \mapsto -1_\gamma$. Hence, we have to enlarge Γ to get every t^g into the image. Via the embedding, we can view Γ as a subset of the totally ordered set $G^{<0}$. We thereby indentify γ with $g = -1_\gamma \in G^{<0}$ and consequently, we can reinterprete (1) as

$$\log_0 t^{-1_g} \ = \ t^g \quad \text{for every } g \in \Gamma \subset G^{<0} \ . \tag{2}$$

We take $\Gamma' := G^{<0}$ to be our enlarged index set. The inclusion $\Gamma \subset \Gamma'$ induces a canonical inclusion $G = \mathbb{R}^\Gamma \subset \mathbb{R}^{\Gamma'} =: G'$. We extend the logarithm to the units of $\mathbb{R}[[G']]$ by the same definition as before. It remains to define $\log_0 t^{g'}$ for $g' \in \mathbb{R}^{\Gamma'} \setminus \mathbb{R}^\Gamma$. We set

$$\log_0 t^{-1_g} \ := \ t^g \quad \text{for every } g \in \Gamma' = G^{<0} \subset G'^{<0} \ . \tag{3}$$

In the same spirit as before we define, for every $g' \in G'$:

$$\text{if } g' = \sum_{g\in G^{<0}} r_g 1_g \ , \ \text{then} \quad \log_0 t^{g'} \ := \ \sum_{g\in G^{<0}} -r_g t^g \ . \tag{4}$$

This shows that every element of $\mathbb{R}((G))$ with support in $G^{<0}$ is in the image of $\log_0$. Since also every element of $\mathbb{R}[[G]]$ is already in the image of $\log_0$ on $\mathbb{R}((G))$, we obtain that every element of $\mathbb{R}((G))$ is in the image of $\log_0$ on $\mathbb{R}((G'))$. But again, if $g' \in G'^{<0} \setminus G$, then $t^{g'}$ is not in the image of $\log_0$.

We will make $\log_0$ surjective by taking the union over an ascending chain of power series fields. We start our construction with an arbitrary totally ordered set Γ_0 and the ordered group $G_0 := \mathbb{R}^{\Gamma_0}$ in the place of Γ and G. Having constructed Γ_n and $G_n = \mathbb{R}^{\Gamma_n}$, we set $\Gamma_{n+1} := \Gamma'_n = G_n^{<0} \supset \Gamma_n$ and $G_{n+1} := G'_n = \mathbb{R}^{\Gamma_{n+1}} \supset G_n$ and extend $\log_0$ as given by (3) and (4). Finally, we set $K := \bigcup_{n\geq 0} \mathbb{R}((G_n))$. Since every element of $\mathbb{R}((G_n))$ lies in $\log_0(\mathbb{R}((G_{n+1})))$, we find that $\log_0$ is surjective on K. The restricted analytic structure of every $\mathbb{R}((G_n))$ carries over to the union. With this structure, K is a model of the theory of the reals with restricted analytic functions.

If we define $G := \bigcup_{n\geq 0} G_n$, then $K \subset \mathbb{R}((G))$. In particular, $t^g \in K$ for every $g \in G$. But K is not itself a power series field. In fact, it cannot be, in view of the result of [K–K–S].

Finally, let us note that for $g, h \in G$,

$$g \ll h < 0 \ \Rightarrow \ -1_g \ll h < 0 \ . \tag{5}$$

(We write $g \ll h < 0$ if $g < h < 0$ and $\forall n \in \mathbb{N} : g < nh$.) To see this, assume that $g, h \in G_n = \mathbb{R}^{\Gamma_n}$, write $h = \sum_{\gamma\in\Gamma_n} r_\gamma 1_\gamma$ and let γ_0 be minimal with $r_{\gamma_0} \neq 0$. Then h is archimedean equivalent to -1_{γ_0}, and $g \ll -1_{\gamma_0} < 0$. Therefore, $g < \gamma_0$ in Γ_{n+1} and thus, $-1_g \ll -1_{\gamma_0} < 0$ in G_{n+1}. Hence also $-1_g \ll h < 0$ in G_{n+1}.

2 Logarithms with appropriate growth rate

Suppose that we have a map exp on K which is an isomorphism from the additive group onto the positive multiplicative group of K and whose restriction to $[-1,1]$ makes K into a model of restricted real exponentiation. Then by Ressayre's Theorem, exp makes K into a model of real exponentiation if and only if it satisfies the axiom

$$\textbf{(GA)} \qquad z > m^2 \implies \exp z > z^m \qquad (m \in \mathbb{N})\,.$$

In [D–M–M1], Ressayre's Theorem is extended as follows: if K is a model of the reals with restricted analytic functions, if $\exp|_{[-1,1]}$ coincides with the restricted analytic function corresponding to the natural exp of the reals, and if exp satisfies (GA), then $(K, \exp)$ is a model of the reals with restricted analytic functions and (unrestricted) exponentiation. We need the following valuation theoretical interpretation of (GA). Its easy proof is given in [K–K1].

Lemma 2.1 *With the assumptions on exp as above, axiom (GA) is satisfied if and only if for all positive infinite z, $\exp z > z^m$ for all $m \in \mathbb{N}$. This in turn is equivalent to*

$$vz < 0 \wedge z > 0 \Rightarrow vz < v\log z < 0\,. \tag{6}$$

We have that $vt^{-1_g} = -1_g$ and $v\log_0 t^{-1_g} = vt^g = g$. Let g_0 be the smallest element in the support of g. If $g < -1_{g_0}$, then $-1_g \ll -1_{g_0} < 0$ and thus, $-1_g \ll g < 0$. If $g > -1_{g_0}$, then $-1_{g_0} \ll -1_g < 0$ and thus, $g \ll -1_g < 0$. This shows that $\log_0$ does not satisfy criterion (6).

To remedy this deficiency, we start our construction with a totally ordered set Γ_0 which admits at least one increasing order automorphism σ. (By this we mean an order automorphism σ such that $\sigma\gamma > \gamma$ for all $\gamma \in \Gamma$.) For the most basic construction, take $\Gamma_0 = \mathbb{Z}$ and $\sigma\gamma := \gamma + 1$.

Every increasing order automorphism σ of a totally ordered set Γ lifts in a canonical way to an order automorphism

$$\sum_\gamma r_\gamma 1_\gamma \mapsto \sum_\gamma r_\gamma 1_{\sigma\gamma}$$

of $\mathbb{R}^\Gamma$. We note that

$$\sum_\gamma r_\gamma 1_\gamma < 0 \quad \text{implies that} \quad \sum_\gamma r_\gamma 1_\gamma < 0 \ll \sum_\gamma r_\gamma 1_{\sigma\gamma} < 0\,.$$

In particular, the induced automorphism is an increasing order automorphism of $(\mathbb{R}^\Gamma)^{<0}$. It extends the automorphism induced by $-1_\gamma \mapsto -1_{\sigma\gamma}$ on the image of Γ in $(\mathbb{R}^\Gamma)^{<0}$. Hence by induction on n, every increasing order automorphism of Γ_0 induces an increasing order automorphism of Γ_{n+1} extending the one on Γ_n. We obtain an increasing order automorphism σ of $\bigcup_{n\geq 0}\Gamma_n = \bigcup_{n\geq 0} G_n^{<0} = (\bigcup_{n\geq 0} G_n)^{<0} = G^{<0}$. Now we define

$$\log_\sigma t^{g'} := \log_0 t^{\sigma g'}\,. \tag{7}$$

Since σ is strictly increasing, it follows that $-1_\gamma \ll -1_{\sigma\gamma} < 0$. Hence,

$$\text{if } g' = \sum_g r_g 1_g < 0 \text{ , then } g' \ll \sigma g' = \sum_g r_g 1_{\sigma g} < 0 \tag{8}$$

and

$$\text{if } g' = \sum_g r_g 1_g \text{ , then } \log_\sigma t^{g'} = \sum_g -r_g t^{\sigma g} \, . \tag{9}$$

Let g_0 be the minimal element in the support of g'. Then $vt^{g'} = g'$ is archimedean comparable to -1_{g_0}. On the other hand, $v \log_\sigma t^{g'} = v\sum_g -r_g t^{\sigma g} = vt^{\sigma g_0} = \sigma g_0$. By (8), $g_0 \ll \sigma g_0 < 0$. By (5), it follows that $-1_{g_0} \ll \sigma g_0 < 0$, and therefore also $vt^{g'} \ll v \log_\sigma t^{g'} < 0$. This proves that $\log_\sigma$ satisfies criterion (6). Hence, $\exp_\sigma := \log_\sigma^{-1}$ makes K into a model of real exponentiation with restricted analytic functions.

• Construction with $\Gamma_0 = \mathbb{Z}$

If we start with $\Gamma_0 = \mathbb{Z}$ and $\sigma(z) = z + 1$, then it follows by induction on the Γ_n that the sequence $\sigma^k\gamma$, $k \in \mathbb{N}$, is cofinal in $\bigcup_n \Gamma_n$ for every $\gamma \in \bigcup_n \Gamma_n$. This yields that for every positive infinite element $a \in K$, the sequence $\log_\sigma^k a$, $k \in \mathbb{N}$, is coinitial in the set of positive infinite elements. This in turn yields that for every positive infinite element $a \in K$, the sequence $\exp_\sigma^k a$, $k \in \mathbb{N}$, is cofinal in K. In other words, $(K, \exp_\sigma)$ has exponential rank 1, in the sense of [K–K1]. For later use, we denote this model by $\mathbb{R}((t))^{EL}$ and its exponential and logarithm simply by exp and log. Let us note that by construction,

$$\gamma \in \Gamma_0 \Rightarrow \log t^{-1_\gamma} = t^{-1_{\gamma+1}} \, . \tag{10}$$

• Construction with other Γ_0

We have shown that *every* increasing order automorphism of Γ_0 gives rise to a logarithm with an appropriate growth rate. Essentially different order automorphisms will yield essentially different growth rates. In fact, the order automorphism σ determines the exponential rank of $(K, \exp_\sigma)$, cf. [K–K1]. For example, the lexicographic product $\Gamma_0 = \mathbb{Z} \times \mathbb{Z}$ has two essentially different increasing order automorphisms: $\sigma_1 : (m,n) \mapsto (m+1, n)$ and $\sigma_2 : (m,n) \mapsto (m, n+1)$. While the sequence $\sigma_1^k(m,n)$, $k \in \mathbb{N}$, is cofinal in Γ_0 for every $(m,n) \in \Gamma_0$, this is never the case for σ_2. It turns out that $(K, \exp_{\sigma_1})$ has exponential rank 1, while $(K, \exp_{\sigma_2})$ has exponential rank $\mathbb{Z}$. However, both are models of the reals with exponentiation. We conclude: there are real closed fields which can be models of the reals with exponentiation, for two or more exponentials of distinct exponential rank. It is not difficult to provide ordered sets Γ_0 with any fixed cardinality of essentially different increasing order automorphisms. So there is no limit on the number of distinct exponential ranks that can be realized at the same time.

Let us add an even more intriguing observation. Even if we start with $\Gamma_0 = \{1\}$, we can obtain countably infinitely many exponentials with distinct exponential rank which turn K into a model of real exponentiation. Indeed, in modification of our above construction, one can let Γ_n play the role of Γ_0 and construct $\log_0$ on $\mathbb{R}((G_n))$. There, we already have the index set Γ_n at hand, which admits at least n essentially different increasing order automorphisms. We leave the details to the reader.

Finally, we note:

Remark 2.2 On the last page of their paper [D–S], van den Dries and Speisegger mention that there is a natural way to expand the power series field $\mathbb{R}((G))$, for divisible G, into a structure for all convergent generalized power series. Since K is just the union over the ascending chain of power series $\mathbb{R}((G_n))$, all G_n divisible, this structure carries over to K.

3 Truncation closed embeddings

In [K–K2], we showed that truncation closed embeddings of Hardy fields in logarithmic-exponential power series are not needed to derive the main results of the paper [D–M–M2]. Nevertheless, we wish to show in this section how a truncation closed embedding of $H(\mathbb{R}_{\mathrm{an,exp}})$ in $(\mathbb{R}((t))^{EL}, \exp)$ can be achieved. More generally, we will show this for all of its subfields $LE_{\mathcal{F}}(x)$, where $x \in H(\mathbb{R}_{\mathrm{an,exp}})$ is positive infinite and $\mathcal{F}$ is chosen as in [K–K2]. See [K–K2] for the notation and the properties of $LE_{\mathcal{F}}(x)$.

According to Theorem 4.6 of [K–K2], we write $LE_{\mathcal{F}}(x)$ in the form $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$ with vx_i, $i \in I$, rationally independent, and with $\log^m x$, $m \geq 0$, among the x_i. Now we choose some $\gamma \in \Gamma_0$ and send x to t^{-1_γ}. By Lemma 3.1 a) of [D–M–M2], the image $\mathbb{R}(t^{-1_\gamma})$ of $\mathbb{R}(x)$ in $\mathbb{R}((t))^{EL}$ is truncation closed. Further, (10) shows that $\log^m x$ has to be sent to $t^{-1_{\gamma+m}}$. By induction on m and Lemma 3.1 a) of [D–M–M2], it follows that the image $\mathbb{R}(t^{-1_{\gamma+m}} \mid m \geq 0)$ of $\mathbb{R}(\log^m x \mid m \geq 0)$ in $\mathbb{R}((t))^{EL}$ is truncation closed.

By Lemma 3.1 b) and Lemma 3.3 of [D–M–M2] we know that if $L \subset H(\mathbb{R}_{\mathrm{an,exp}})$ is already embedded in $\mathbb{R}((t))^{EL}$ with a truncation closed image, then also $L^{\mathrm{r}\mathcal{F}}$ has truncation closed image in $\mathbb{R}((t))^{EL}$. Therefore, it only remains to consider the images for the remaining x_i's. Recall that our construction given in [K–K2] leaves us some freedom in the choice of the x_i's. In particular, if x_i comes in as $\exp a$, with a in an already constructed r$\mathcal{F}$-closed field L, but $\exp a \notin L$, and if $a = a' + b$ with $vb \geq 0$, then $\exp b \in L$ and thus, we can replace x_i by $\exp a'$. Assume that we have already an embedding ι of L in $\mathbb{R}((t))^{EL}$, with truncation closed image ιL. Then $\exp \iota a \notin \iota L$. Now ιa is a power series, and we can write $\iota a = a'_0 + b_0$ where $vb_0 \geq 0$ and a'_0 is a power series with negative support. Since ιL is truncation closed, $a'_0, b_0 \in \iota L$, and we can write $a = a' + b$ with $a'_0 = \iota a'$, $b_0 = \iota b$ and $vb \geq 0$. By what we said above, we may replace x_i by $\exp a'$. This has to be sent to $\exp a'_0$. Since a'_0 is a power series with negative support, $\exp a'_0$ is of the form t^g, by construction. As before, we conclude that the image $(\iota L)(t^g)^{\mathrm{r}\mathcal{F}}$ of $L(x_i)^{\mathrm{r}\mathcal{F}}$ is truncation closed. By transfinite induction on the x_i's, one now obtains an embedding of $LE_{\mathcal{F}}(x) = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$ in $\mathbb{R}((t))^{EL}$ with truncation closed image.

Let us note:

Remark 3.1 As in [D–M–M2], let $i(x)$ denote the compositional inverse of the function $x \log x$. Since ιx lies in the first field $\mathbb{R}((G_0))$ of our ascending chain of power series fields, so does $\iota i(x)$. This follows from the expression derived for $i(x)$ in the proof of Theorem 4.8 of [D–M–M2], and the fact that already $\mathbb{R}((G_0))$ is r$\mathcal{F}_{an}$-closed and closed under log.

4 Derivations

We wish to show how to endow $(\mathbb{R}((t))^{EL}, \exp)$ with a derivation D. Again in the spirit of "ultrametric continuity", we want that

$$D \sum_g r_g t^g = \sum_g r_g D t^g . \tag{11}$$

Since moreover Dt^{-g} is uniquely determined by Dt^g, it thus suffices to define Dt^g for all $g \in G^{<0}$. Further, if $a, b \in K$ such that $a = \log b$, then Db is uniquely determined by Da, and conversely. In view of (4), (3) and (11) it thus suffices to define Dt^{-1_g} for all $g \in G^{<0}$. But if $-1_g \in G_{n+1}$ then $g \in G_n$ and $\log t^{-1_g} = t^{\sigma g}$ with $\sigma g \in G_n$. Hence to define Dt^{-1_g} for $-1_g \in G_{n+1}$, it suffices to define it for $-1_g \in G_n$. By induction, we find that it suffices to define Dt^{-1_γ} for $\gamma \in \Gamma_0 = \mathbb{Z}$. In view of (10), it suffices to define Dt^{-1_γ} for a single $\gamma \in \Gamma_0 = \mathbb{Z}$. We pick one, say γ_0, and set $Dt^{-1_{\gamma_0}} = 1$. This determines the derivation D on K uniquely.

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