Nested Pfaffian functions

(Joint with Gareth and Siegfried)

Motivation: when you unravel Khovanskii's original definition of "Pfaffian function", you obtain something seemingly more general than the most commonly found notion of Pfaffian function.

Recap: Pfaffian functions

Let $U \subseteq \mathbb{R}^n$ be open and

$$f = (f_1, \dots, f_k) : U \longrightarrow \mathbb{R}^k$$

be real analytic.

<u>Definition</u>: f is a Pfaffian chain if, for $i = 1, \dots, k$ and $j = 1, \dots, n$, there exist polynomials $P_{ij} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_i]$ such that, for $x \in U$,

(I) $$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \dots, f_i(x)).$$

Example

$f = (e_1, \dots, e_k)$, where e_i denotes the i^{th} compositional iterate of exp; here $U = \mathbb{R}$.

Variants

① Allow partially defined f to include meromorphic functions, roots

② f is pfaffian over an o-min exp. $\mathcal{R}$ of the real field, if the P_{ij} are definable in $\mathcal{R}$.

Definition: $g: U \to \mathbb{R}$ is pfaffian if there is a pfaffian chain $f: U \to \mathbb{R}^k$ such that $g = f_k$.

Set $\mathcal{P}_n :=$ set of germs at 0 in $\mathbb{R}^n$ of all pfaffian functions on nbhds of 0.

For $g \in \mathcal{P}_n$ denote by $\hat{g} \in \mathbb{R}[[x_1, \dots, x_n]]$ its Taylor series at 0.

Proposition: $\mathcal{P}_n$ is an $\mathbb{R}$-algebra, and $g \in \mathcal{P}_n$ is a unit iff $\hat{g}$ is a unit.

$\mathbb{R}_{\text{Pfaff}}$ = expansion of the real field by all global pfaffian functions

Facts

① $\mathbb{R}_{\text{Pfaff}}$ is o-minimal (Wilkie)

② $\mathbb{R}_{\text{Pfaff}}$ is a reduct of the pfaffian closure $\mathcal{P}(\mathbb{R})$ of the real field, and $\mathcal{P}(\mathbb{R})$ is model complete in the language of nested Rolle leaves (Khovanskii's setting).

<u>Question 1:</u> Is $\mathbb{R}_{\text{Pfaff}}$ model complete?

Unravelling Khovanskii's definition

Locally at each point of a nested Rolle leaf, the latter has the following representation:

Fix: • $0 < k < n$

• $I_i = (a_i, b_i)$ with $a_i < 0 < b_i$

• $B_i := I_1 \times \cdots \times I_i$, $B^i := I_{n-i+1} \times \cdots \times I_n$

• $f_i : B_{n-i} \to I_{n-i+1}$ analytic, $i = 1, \dots, k$

and set $f = (f_1, \dots, f_k)$.

<u>Note:</u> $gr(f_i) \subseteq B_{n-i+1}$ for each i.

Notation: for $i = 1, \dots, k-1$, define

$$f_i \circ f_{i+1} : B_{n-i+1} \to I_{n-i}$$

by

$$(f_i \circ f_{i+1})(\bar{x}) := f_i(\bar{x}, f_{i+1}(\bar{x})).$$

Iterating this, for $1 \le j \le i \le n$ we define $f_j \circ \dots \circ f_i : B_{n-i} \to I_{n-j+1}$ by

$$f_j \circ \dots \circ f_i := \begin{cases} f_i & \text{if } j = i \\ (f_j \circ \dots \circ f_{i-1}) \circ f_i & \text{if } j < i. \end{cases}$$

Set

$$\varphi_i := (f_i, f_{i+1} \circ f_i, \dots, f_1 \circ \dots \circ f_i),$$

and for $P \in \mathbb{R}[x_1, \dots, x_n]$, set

$$P \circ \varphi_i(\bar{x}) := P(\bar{x}, \varphi_i(\bar{x})).$$

Definition

The tuple $f = (f_1, \dots, f_k)$ is a nested pfaffian chain if there exist polynomials P_{ij} such that

$$(\mathrm{II}) \qquad \partial_j f_i = P_{ij} \circ \varphi_i .$$

Main examples: Let $g = (g_1, \dots, g_k) : B_{n-k} \to \mathbb{R}^k$ be a pfaffian chain.

Set $f_i : B_{n-i} \to I_{n-i+1}$ as $f_i := g_i$.

Then $\varphi_i = (g_1, \dots, g_i)$, so (I) for g is the same as (II) for $f = (f_1, \dots, f_k)$.

Question 2: Is every np chain pfaffian?

Towards a counterexample

Let $j:(0,\infty)\to\mathbb{R}$ be the function defined by

$$j(t) := j(it),$$

where j is the Klein j-function.

Lemma: There is an open interval $J \subseteq (0,\infty)$ such that $j\restriction_J$ is the compositional inverse of a pfaffian function.

Question 3: Is j itself pfaffian?

The answer to this question is affirmative if the algebras P_n are closed under taking implicit functions. Are they?

Turning things around: if we can show that j is not pfaffian, it follows that the P_n are not closed under taking implicit functions.

<u>Question 4:</u> How do we show some function is not pfaffian?

Fact (Freitag ~2021, based on Freitag & Scanlon)

j is not pfaffian, for any open interval $J \subset (0, \infty)$.

The proof of this fact uses the model theory of differentially closed fields. (See Jim's talk)

Corollary: The algebras $\mathcal{P}_n$ are not closed under taking implicit functions.

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Back to nested pfaffian functions: are their algebras of germs at 0 closed under taking implicit functions?

Almost:

__Re-Definition__

The tuple $f = (f_1, \dots, f_k)$ is a nested pfaffian chain if there exist polynomials P_{ij} and Q_{ij} such that $Q_{ij} \circ \varphi_i \neq 0$ and

$$(\mathrm{II}) \qquad \partial_j f_i = \frac{P_{ij}}{Q_{ij}} \circ \varphi_i$$

Nested pfaffian function := member of a nested pfaffian chain

Proposition

The sets NP_n of germs at 0 of (re-defined) nested pfaffian functions have all the closure properties of the algebras P_n, and they are closed under taking implicit functions.

Proof: Let $f = (f_1, \ldots, f_k)$ be a nested pfaffian chain on the box B, and assume that $\frac{\partial f_k}{\partial x_{n-k}}(c) \neq 0$ while $f_k(c) = 0$.

Let $g: B_{n-k-1} \to \mathbb{R}$ be the corresponding implicit function (after shrinking B).

Then $f_k \circ g = 0$, so for $j = 1, \dots, n-k-1$,

$$0 = \partial_j (f_k \circ g) = (\partial_j f_k) \circ g + (\partial_{n-k} f_k) \circ g \cdot \partial_j g$$
$$= (P_{kj} \circ e_k) \circ g + (P_{k,n-k} \circ e_k) \circ g \cdot \partial_j g.$$

Hence $(f_1, \dots, f_k, g)$ is a nested pfaffian chain.

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<u>Corollary</u>: g is nested pfaffian.

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Question 5: are the algebras $\mathcal{N}_n$ closed under Weierstrass preparation?

Question 5 follows from

Conjecture: If f is a symmetric nested pfaffian function, there exists a nested pfaffian function g such that $f = g \circ \sigma$.

Where:
- f symmetric if $f \circ \pi = f$ for every permutation π of the variables
- $\sigma = (\sigma_1, \ldots, \sigma_n)$ are the elementary symmetric polynomials

Theorem (not quite the conjecture)

Let $f = (f_1, \dots, f_k)$ be a npf chain, and assume that

$$\varphi_k = (f_k, f_{k-1} \circ f_k, \dots, f_1 \circ \dots \circ f_k)$$

is symmetric. Then there exists a npf chain $g = (f_1, \dots, f_{k-1}, g_k)$ such that $f_k = g_k \circ \sigma$.

Proof: Commutative Algebra.

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Are nested pfaffian functions the simplest extension of pfaffian functions closed under taking implicit functions?

Maybe not: let $p_1, \dots, p_k \in \mathbb{N}$ and $q_1, \dots, q_k \in \mathbb{N}$, and for each $i = 1, \dots, k$, $j = 1, \dots, p_i$ and $\ell = 1, \dots, q_i$, let

$$f_{ij} : B_{n-i} \longrightarrow \mathbb{R}$$

drop in # of variables

$$g_{i\ell} : B_{n-i-1} \longrightarrow \mathbb{R}$$

be such that

polynomial

$$\partial_x f_{ij} = P_{ij} \circ (f_{i1}, \dots, f_{ip_i}, g_{i-1,1}, \dots, g_{i-1,q_{i-1}})$$

and each $g_{i\ell}$ is obtained from one of the f_{ij} by the implicit function theorem.

Such a chain looks like this:

$$(\underbrace{f_{11},\dots,f_{1p_1}}_{\text{pf chain}},\underbrace{g_{11},\dots,g_{1q_1}}_{\text{implicit}},\underbrace{f_{21},\dots,f_{2p_2}}_{\text{pf chain}},\underbrace{g_{21},\dots,g_{2q_2}}_{\text{implicit}},\dots)$$

in over in ...

Call this an implicit pfaffian chain, and call any function in such a chain implicit pfaffian.

Then: • j is implicit pfaffian.

• implicit function of ipf functions are ipf.

• Since npf functions are closed under taking implicit functions, every ipf function is npf
$\Longrightarrow$ Khovanskii Theory applies.

No idea if our symmetric function theorem for npf functions works for ipf functions