Purely transcendental extensions of formally real fields have the independence property

Universität Konstanz

Lasse Vogel
joint work in progress with
Lothar Sebastian Krapp, Salma Kuhlmann

Abstract There is a conjecture by Shelah suggesting that infinite fields without the independence property posses distinct algebraic properties. This conjecture has been adapted numerous times, for example [1, main open problems, p.820] or [3, Conjecture 1.9]. Notably every purely transcendental extension of a formally real field fails to have any of these properties. Therefore, if the conjecture holds, all such fields must have the independence property. We show that for any formally real field K the rational function field $K(X)$ has the independence property in the language of rings. The main ingredient will be the explicit definition of an IP-formula φ and the sets witnessing the independence property in the field $\mathbb{Q}(X)$. After that we will transfer this result to the general case of $K(X)$, making use of the interplay of poles of rational functions and bounded elements with respect to a preordering.

Purely transcendental extensions of formally real fields

Let K be **formally real**, i.e. -1 is not a sum of squares or equivalently K can be endowed with an ordering. A field extension L/K is a **purely transcendental** extension of K if there is an over K algebraically independent subset $\mathcal{B} \subsetneq L$ such that $L = K(\mathcal{B})$. Note that L is formally real. Also, for an $X \in \mathcal{B}$, there exists a formally real field K' such that $L = K'(X)$. Hence we only have to consider fields of the shape $K(X)$.

The independence property

Let $\mathcal{L}$ be a first order language and $\mathcal{T}$ an $\mathcal{L}$-theory with infinite models. We say that $\mathcal{T}$ has the **independence property** (IP) if there exists an $\mathcal{L}$-formula $\varphi(x; y)$ (where x and y can be tuples) such that: In every model $\mathcal{M} \models \mathcal{T}$ and for any $n \in \mathbb{N}$ there is an $A_n \subset M^{\ell(x)}$, where ℓ denotes the length of the tuple x, with $|A_n| = n$ such that for every $I \subseteq A_n$ there exists a tuple b_I with $\ell(b_I) = \ell(y)$ so that for every $a \in A_n$:

$$\mathcal{M} \models \varphi(a; b_I) \Leftrightarrow a \in I.$$

Example. Let $\mathcal{L}_r = \{0, 1, +, -, \cdot\}$ be the of rings and let $\mathcal{T}$ be the theory of some fixed ring R. The $\mathcal{L}_r$-formula

$$\varphi(x; y) := \exists z : xz = y$$

expresses divisibility in a ring. Let $A_n = \{a_1, \dots, a_n\}$ where the a_i are non-related prime elements of R. For $I \subseteq A_n$ define $b_I := \prod_{a \in I} a$. For $a \in I$ we have $a \mid b_I$, but for $a' \in A_n \setminus I$ we have $a' \nmid b_I$. Hence if there is an *infinite* set of non-related primes, we can choose any subset of cardinality n as A_n for any $n \in \mathbb{N}$. As a result $\mathcal{T}$ has the independence property.

In particular this works for the polynomial ring $K[X]$ over any infinite field K, as the set $\{X - a \mid a \in K\} \subsetneq K[X]$ consists of non-related primes.

On poles of rational functions

We modify the approach above in order to obtain that the theory of $K(X)$ has IP. For this we change the formula φ to express "divisibility" in a definable subset of $K(X)$. The problematic factors for the example set from the block above are the functions of the form $\frac{1}{X-a}$ for $a \in K$. We observe that each such function has a pole at a. For a $\mathcal{P} \subsetneq K(X)$ consisting only of $f \in \mathcal{P}$ without poles and $g_1, g_2 \in \mathcal{P}$ we obtain that $\frac{g_1}{g_2} \in \mathcal{P}$ implies that every root of g_2 is already a root of g_1.

To define such a set $\mathcal{P}$, we make use of the following:

Lemma. *Let K be a formally real field and let $f \in K(X)$ with a pole at $a \in K$. Then for any ordering $<$ on K and $u \in K$, there is a $b \in K$ such that $|f(b)| > |u|$.*

This lemma is proved via observation of $\frac{1}{f}$ in a real closure of K and using continuity of rational functions with respect to the order topology.

As a result of this lemma, every function with at least one pole is unbounded on a neighbourhood of every pole with respect to any ordering on the field K. It follows that functions with bounded values are pole-free. We will use this to define the set $\mathcal{P}$.

Contact

Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz, Germany
lasse.vogel@uni-konstanz.de

Bounded rational functions over $\mathbb{Q}$

Given an ordered field $(K, <)$. A function $f : K \to K$ is called **bounded** if there exists $b \in K$ such that for every $a \in K$ it is $-b < f(a) < b$. Note that a function f is bounded if and only if there exists a $c \in K$ such that $c - f^2$ is **positive semidefinite**. Describing positive semidefinite rational functions is one of the main problems of real algebra, in the case where $K = \mathbb{Q}$ we can make use of the following results:

Fact. *(cf. [2] Main Theorem) Every univariate positive semidefinite rational function over $\mathbb{Q}$ is the sum of 5 squares.*

$\mathbb{Q}(X)$ has the independence property

Theorem. *The formula*

$$\varphi(x; y) := \left(\exists z, u_1, u_2, u_3, u_4, u_5 : \sum_{i=1}^{5} u_i^2 = 1 - z^2 \land x \cdot z = y \right)$$

witnesses the independence property of the theory of $\mathbb{Q}(X)$.

$\varphi(x; y)$ basically states (in $\mathbb{Q}(X)$) that $\frac{y}{x}$ is bounded by 1, hence $\frac{y}{x}$ is pole-free. In particular if x has a root at some $a \in \mathbb{Q}$, then y must also have a root at a. Working with bounded by 1 instead of bounded by some arbitrary constant is done to save the effort of having to define constants in the function field. The specific bound 1 is chosen, as it transfers to products of functions bounded by 1.

The sets A_n now consist of the following elements: For some $i \in \mathbb{N}$ consider

$$p_i := \frac{X - i}{(i+1)(X^2+1)}.$$

Each such p_i is bounded by 1 and has a singular root at $i \in \mathbb{Q}$. Now set

$$A_n := \{p_i \mid 1 \leq i \leq n\}$$

for every $n \in \mathbb{N}$ and for $I \subseteq A_n$ define $b_I := \prod_{p \in I} p$. All b_I are also bounded by 1 and for $p \in I$ it is $\frac{b_I}{p} = b_{I \setminus \{p\}}$, hence $\mathbb{Q}(X) \models \varphi(p, b_I)$. On the other hand if $p_i \notin I$, then b_I does not have a root at i, hence $\frac{b_I}{p_i}$ has a pole at i and thus $\mathbb{Q}(X) \not\models \varphi(p_i, b_I)$. This suffices to show that the theory of $\mathbb{Q}(X)$ has the independence property.

Transfer to any formally real rational function field

We now want to show that the same formula $\varphi(x; y)$ witnesses IP of the theory of any formally real rational function field $K(X)$, and by extension that $K(X)$ has the independence property. First note that $\mathbb{Q}(X)$ can be embedded into $K(X)$. As a result we can also consider the p_i and b_I as elements of $K(X)$ and it only remains to show that

$$K(X) \models \varphi(p_i, b_I) \Leftrightarrow \mathbb{Q}(X) \models \varphi(p_i, b_I).$$

Since $\varphi(x; y)$ is an existential formula and $\mathbb{Q}(X) \subseteq K(X)$, it follows already that $\mathbb{Q}(X) \models \varphi(p_i, b_I) \Rightarrow K(X) \models \varphi(p_i, b_I)$.

On the other hand if $\mathbb{Q}(X) \not\models \varphi(p_i, b_I)$, then $\frac{b_I}{p_i}$ has a pole at i. Thus it is unbounded on a neighbourhood of i and in particular there is $a \in K$ such that $|p_i(a)| > 1$. Hence $1 - (\frac{b_I}{p_i})^2(a) < 0$ and $1 - (\frac{b_I}{p_i})^2$ is not positive semidefinite. But every sum of squares over a formally real field has to be positive semidefinite with respect to any ordering. As a result we obtain $K(X) \not\models \varphi(p_i; b_I)$.

With this the desired result has been established, every purely transcendental extension of a formally real field has the independence property.

Open questions: Algebraic extensions

We have seen that for formally real fields a purely transcendental extension always causes the independence property. Next is to examine how the independence property interacts with algebraic extensions. We can already say:

(a) Algebraic extensions of formally real fields with IP may not have IP, for example any real closure.
(b) Finite algebraic extensions of formally real fields without IP again can not have IP (shown by interpretability of the extension in the original field).

The questions I am now interested in are

(i) Let K be a formally real field with IP and L/K a finite real algebraic extension. Does L always have IP?
(ii) Let K be a formally real field without IP and L/K a real algebraic extension. Does L necessarily not have IP?

References

[1] K. Dupont, A. Hasson and S. Kuhlmann, 'Definable valuations induced by multiplicative subgroups and NIP fields', *Arch. Math. Logic* 58 (2019) 819–839.
[2] J.S. Hsia and R.P. Johnson, 'On the Representation in Sums of Squares for Definite Functions in One Variable Over an Algebraic Number Field', *Amer. J. Math.* 96 (1974) 448–453.
[3] W. Johnson, 'The classification of dp-minimal and dp-small fields', *J. Eur. Math. Soc.* 25 (2023) 467–513.

Acknowledgement

This work is part of my doctoral research project, which started in 2022 and is supervised by Professor Salma Kuhlmann and mentored by Dr. Lothar Sebastian Krapp at Universität Konstanz.