Domination and Fibres

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Abstract

In stable theories, we can understand types of finite rank in terms of a finite collection of minimal types. One such method is a domination decomposition of the type. Consider a stationary type $p \in S(A)$ over a small parameter set A and some fibration $f : p \to f(p)$. We are interested in whether or not p is domination equivalent to the Morley product of its fibre $p_{f(a)}$ and image $f(p)$. In particular, we show that when the fibres are almost internal to a minimal type r we have that $p \square f(p) \otimes r^{(m)}$ for $0 \leq m \leq U(p_{f(a)})$ and $m = U(p_{f(a)})$ if and only if the fibre is uniformly relatively r-internal.

Introduction

We work in a stable theory T of finite rank. Fix $\mathcal{U}$ a sufficiently saturated model of T. All parameter sets A are considered to be small with respect to the saturation of $\mathcal{U}$. We assume that up to non-orthogonality there is a unique non-locally modular minimal type over $\emptyset$. We assume that all types p are stationary unless otherwise stated.

Definition 1: Domination

Let $p \in S(A)$ and $q \in S(B)$ be two types. We say that p **dominates** q denoted $p \trianglerighteq q$ if for some set $C \supset A \cup B$ there are $a \models p|_C$ and $b \models q|_C$ such that for all sets d, if $a \mathop{\perp\!\!\!\perp}_C d$ then $b \mathop{\perp\!\!\!\perp}_C d$.

We say p is domination equivalent to q, denoted $p \square q$ if $p \trianglerighteq q$ and $q \trianglerighteq p$.

Lemma 1: Domination of Images

Let $p, q \in S(A)$, $p = \text{tp}(a/A)$ and $f : p \to q$ be a definable map. Then p dominates $\text{tp}(f(a)/A)$.

Let $p \in S(A)$ be a type. Then p is domination equivalent to a finite product of minimal types ([3]). Moreover the set of minimal types appearing in the decomposition are unique up to non-orthogonality. Under the assumption that there is a unique non-locally modular minimal type up to the non-orthogonality, we may choose $r_1, \ldots, r_n \in S(A)$. The following shows that it is sufficient to take $C = M$, an $\aleph_1$-saturated model and d to be such that $\text{tp}(d/M)$ is minimal in the definition of domination.

Lemma 2: Minimal Domination

Let $p \in S(A)$ and $q \in S(B)$ be finite U-rank types. Then $p \trianglerighteq q$ if and only if for some (any) $\aleph_1$-saturated model $M \supset A \cup B$, there exist $a \models p|_M$ and $b \models q|_M$ such that for all e with $\text{tp}(e/M)$ minimal, we have that $a \mathop{\perp\!\!\!\perp}_M e \Rightarrow b \mathop{\perp\!\!\!\perp}_M e$.

Domination and fibrations

Let $p, q \in S(A)$ be types. We say $f : p \to q$ is a **fibration** if it is an A-definable map such that for all $a \models p$ with $f(a) \models q$ and $\text{tp}(a/f(a)A)$ is stationary. Let $p_{f(a)} = \text{tp}(a/Af(a))$ denote the fibre of the map f at the point $f(a)$.

Lemma 3

Let $p \in= S(A)$ and $f : p \to f(p)$ be a fibration. If $p \perp \text{tp}(a/f(a)A)$ for any $f(a) \models f(p)$, then $p \square f(p)$.

As a consequence we get that p need not dominate its fibres.

Example 1: DCF$_0$

In DCF$_0$ the unique non-locally modular minimal type r is the generic type of the constants $(\mathcal{C})$.

Consider the generic type p of the differential algebraic variety given by $y' = xy, x' = x^3(x-1)$, and the fibration f given by projection onto the x-coordinate. It is shown in [1, Corollary 5.5] that $p \perp \mathcal{C}$.

But for any $b \models p$, $p_{f(b)}$ is internal, and thus nonorthogonal, to $\mathcal{C}$.

Thus $p \perp p_{f(b)}$ for all $b \models p$, and the above shows that $p \square f(p)$.

Orthogonality with fibres is not the only way for a type to be dominated by its image.

Example 2: DCF$_0$

Consider $p \in S(\emptyset)$ the generic type of $(\frac{x'}{x})' = 0$, i.e., this type is the generic type of $\delta\log^{-1}(\mathcal{C})$, where $\delta\log$ is the logarithmic derivative. In this case, $r = \delta\log(p)$ is the generic type of the constants, the fibres are internal to the constants (hence $p \not\perp \text{tp}(a/\delta\log(a))$) and $p \square r$.

A type $p = \text{tp}(a/B)$ has **no proper fibrations** if for all $b \in \text{dcl}(aB) \setminus \text{acl}(B)$ we have $a \in \text{acl}(bB)$. If $f : p \to q$ is a fibration (over A), we say that $f : p \to q$ has no proper fibrations if $\text{tp}(a/f(a)A)$ has no proper fibrations [2].

Lemma 4: No proper fibration domination dichotomy

Let $p \in S(A)$ and let $f : p \to f(p)$ be a fibration such that its fibres are almost r-internal, have no proper fibrations, and are of U-rank n. Then either $f(p) \square f(p)$ or $p \square f(p) \otimes r^{(n)}$ (and in that case $f : p \to f(p)$ is uniformly almost r-internal).

Example 3: DCF$_0$

Let q be the generic type of the differential algebraic variety given by $y' = xy$, $x' = x^2(x-1)$, and the fibration f is given by projection onto the x-coordinate. It is shown in [1, Corollary 5.5] that q is non-orthogonal to $\mathcal{C}$ and that $f : q \to f(q)$ is uniformly relatively almost internal to $\mathcal{C}$. Suppose for all $b \models q$, $U(p_{f(b)}) = 1$ thus $q \square f(q) \otimes r$ where r is the generic type of the constants.

Domination and Uniform Almost Internality

Let $p \in S(A)$ and $f : p \to q$ be a fibration. We say that $f : p \to q$ is **uniformly almost** internal to a minimal type r if there is a finite tuple e of parameters such that for some (any) $a \models p$ with $a \mathop{\perp\!\!\!\perp}_A e$, we have $a \in \text{acl}(f(a), e, r(\mathcal{U}))$.

Theorem 1: Domination Decomposition of fibration

Let $p, q \in S(A)$ and $f : p \to q$ be a fibration and let the fibre $\text{tp}(a/f(a)A)$ be almost internal to a minimal type $r \in S(B)$ for all $a \models p$. Then $p \square f(p) \otimes r^{(m)}$ for some $0 \leq m \leq U(a/f(a)A) = n$. Moreover, $m = U(a/f(a)A)$ if and only if $f : p \to f(p)$ is uniformly almost internal to r.

Proof 1: (sketch)

If $\text{tp}(f(a)/A) \trianglerighteq p$ we set $m = 0$ and we are done.

Otherwise, we show that the witness to $f(p) \ntrianglerighteq p$ can be chosen to be an independent tuple of r. To do so, we inductively construct some $\aleph_1$-saturated model M, $b \models p|_M$ and $c_1, \ldots, c_k \models r^{(k)}$ such that: (1) $b \mathop{\perp\!\!\!\perp}_A M$, (2) $c_1, \cdots, c_k \mathop{\perp\!\!\!\perp}_M f(b)$ (3) $f(b)c_1 \cdots c_{k-1} \mathop{\perp\!\!\!\perp}_M c_k$ (4) $b \not\mathop{\perp\!\!\!\perp}_{c_1 \cdots c_{k-1} f(b) M} c_k$, (5) $f(b), c_1, \cdots, c_k \in \text{acl}(bM)$ (gives $p \trianglerighteq f(p) \otimes r^{(k)}$).

If $m = n$, uniform almost r-internality follows from $b \in \text{acl}(c_1 \cdots c_n f(b) M)$ and $b \mathop{\perp\!\!\!\perp}_A M$.

Acknowledgements

This research was conducted at the University of Waterloo under the supervision of Rahim Moosa.

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