Quasiminimality of a correspondence between two elliptic curves

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Abstract

It is well-known that the field of complex numbers is minimal. However, adding the exponential map to the structure makes it possible to define the ring of integers, preventing minimality and many other nice properties. Nevertheless, Zilber's quasiminimality conjecture states that the complex exponential field is quasiminimal, i.e. every definable subset is countable or co-countable. Analogous conjectures were made, replacing the exponential map with other interesting analytical functions, including the Weierstrass elliptic functions. We consider a correspondence between two elliptic curves and prove that the emerging structure is quasiminal, thus obtaining a weaker version of such conjecture.

The field of complex numbers

- Let $\mathbb{C}_{\text{field}} = (\mathbb{C}; +, -, \cdot, 0, 1)$.
- Its theory ACF_0 admits quantifier elimination.
- Hence, any formula in one variable is equivalent to a boolean combination of formulas $p(x) = 0$, where p is a polynomial. Note that it has finitely many zeros.
- Therefore, $\mathbb{C}_{\text{field}}$ is minimal: every definable set is finite or co-finite

Complex numbers with the exponential map

- What happens if we add the exponential map and look at $\mathbb{C}_{\exp} = (\mathbb{C}; +, -, \cdot, 0, 1, \exp)$?
- Note that $\ker \exp = 2\pi i\mathbb{Z}$ is definable then.
- Moreover, the multiplicative kernel stabilizer $\{z \in \mathbb{C} \mid \forall a \in \ker : za \in \ker\}$ is definable and equal to $\mathbb{Z}$.
- Therefore, $\mathbb{C}_{\exp}$ interprets first-order arithmetic and is definitely not minimal.

Definition

A structure $\mathcal{M}$ is quasiminimal if every definable subset of $\mathcal{M}$ is countable or co-countable.

Zilber's conjecture

The structure $\mathbb{C}_{\exp}$ is quasiminimal.

Example

The structure $\mathbb{C}_Z = (\mathbb{C}; +, -, \cdot, 0, 1, Z)$, where Z is a predicate for integers, is quasiminimal.

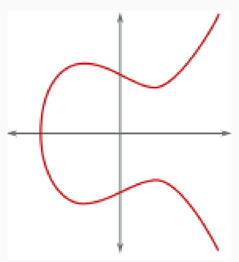
Proof. Suppose a subset $S \subseteq \mathbb{C}$ is definable with parameters $\overline{a}$. Note that for any two numbers s, t transcendental over $\mathbb{Q}(\overline{a})$ there exists a field isomorphism mapping s to t and identical on $\mathbb{Q}(\overline{a})$. Hence, either $\mathbb{C} \setminus \mathbb{Q}(\overline{a})^{\text{alg}} \subseteq S$ or $S \subseteq \mathbb{Q}(\overline{a})^{\text{alg}}$. □

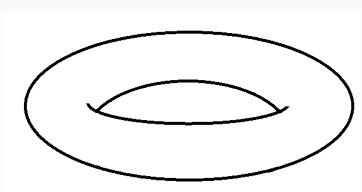
Reducts

- Note that if $\mathcal{M}$ is quasiminimal and $\mathcal{N}$ is a reduct of $\mathcal{M}$, then $\mathcal{N}$ is also quasiminimal.
- In order to approach Zilber's conjecture we can try proving the quasiminimality of reducts of $\mathbb{C}_{\exp}$.
- For example, in [1], it is shown that the a reduct $\mathbb{C}_{\text{AE}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{AE}})$, where $\Gamma_{\text{AE}} = \{(x, e^{x+q+2\pi i r}) : x \in \mathbb{C}; q, r \in \mathbb{Q}\}$, is quasiminimal.

Elliptic curves

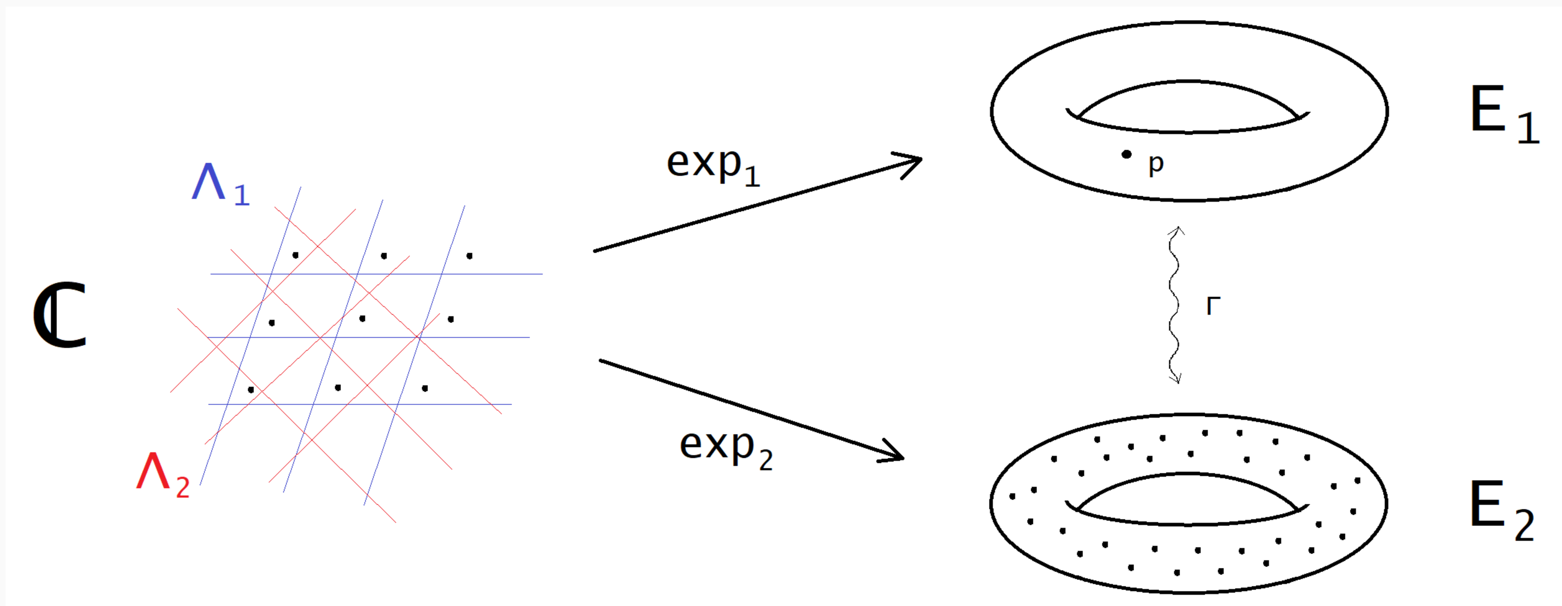
- Let Λ be a lattice on $\mathbb{C}$.
- It induces Weierstrass elliptic function $\wp_\Lambda(z) = \frac{1}{z^2} + \sum\limits_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$.
- Let $\exp_\Lambda : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ be defined by $\exp_\Lambda(z) = \begin{cases} [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1] & \text{if } z \notin \Lambda, \\ [0:1:0] & \text{if } z \in \Lambda. \end{cases}$
- Then the elliptic curve E_Λ is the image of the map $\exp_\Lambda$.
- On the other hand we can define E_Λ as a zero-set of a homogeneous polynomial.
- The map $\exp_\Lambda : \mathbb{C}/\Lambda \to E_\Lambda$ also defines an automorphism of additive groups, turning E_Λ into an algebraic group.





Correspondence between two elliptic curves

- Let E_1 and E_2 be two complex elliptic curves with exponential maps $\exp_1$ and $\exp_2$.
- Let $\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$.
- We show that the structure $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, -, \cdot, 0, 1, \Gamma_{\text{corr}})$ is quasiminimal.



Conjectures for $\wp$

- Let $\{\wp_i : i \in I\}$ be a countable set of Weierstrass elliptic functions.
- Then it is conjectured that a structure $(\mathbb{C}; +, -, \cdot, 0, 1, \{\wp_i : i \in I\})$ is quasiminimal.
- Our result is quasiminimality of a reduct of such a structure with two Weierstrass elliptic functions.

Proof Strategy

The proof is analogous to [1]. Using results from [2], it suffices to show two things:

- countable closure property,
- Γ-closedness.

We discuss them further.

Countable closure property

- Let $\Psi_{\text{corr}} = \{\wp_2 \circ (\wp_1 \restriction_\delta)^{-1} : \delta \in \Delta\}$, where Δ is a countable set of boxes δ with rational coordinates, with $\wp_2 \circ (\wp_1 \restriction_\delta)^{-1}$ well-defined on each δ and Δ covering all of E_1.
- Close Ψ_{corr} under polynomials with integer coefficients.
- Consider a pregeometry on $\mathbb{C}$ based on functions implicitly defined from Ψ_{corr}, denoted by $\tilde{D}$ in [3].
- Countable closure property says that for any finite subset $X \subseteq \mathbb{C}$, $\tilde{D}(X)$ is countable.

Γ-closedness

- Γ-closedness means that for every free and rotund irreducible subvariety V of $E_1^n \times E_2^n$ of dimension n, the set $V \cap \Gamma_{\text{corr}}^n$ is non-empty.
- Here freeness and rotundness are specific restrictions, cutting out degenerate cases.
- The main proof components are the density of the layers of the correspondence and openness of holomorphic maps.
- Then the Ax's Theorem is used to check that the dimension of the intersection works out as intended.

References

[1] Jonathan Kirby. "Blurred complex exponentiation". In: *Selecta Mathematica* 25.5 (2019).

[2] Martin Bays and Jonathan Kirby. "Pseudo-exponential maps, variants, and quasiminimality". In: *Algebra & Number Theory* 12.3 (2018), 493–549.

[3] A.J. Wilkie. "Some local definability theory for holomorphic functions". In: *Model Theory with Applications to Algebra and Analysis*. Vol. 1. Cambridge University Press, 2008, 197–214.