Logarithmic Cross-Sections of Real Closed Fields *

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Abstract

We introduce the notion of a logarithmic cross-section and show that the power series field $\mathbb{R}((G))$ admits such, for a large class of groups G. We discuss necessary and sufficient conditions for the existence of an exponential power series field. We show that it is equivalent to the existence of a certain lexicographic ordering, and construct an ordering that is quite close to the required one. Finally, we show how to use logarithmic cross-sections of power series fields to define surjective logarithms on countable limits.

1 Introduction

This paper is a continuation of the work developed in the papers [KS] and [K–K]. However, we shall now concentrate mainly on exponential fields having the same elementary properties as $(\mathbb{R}, \exp)$. We shall make intensive use of Ressayre's axiomatization of the theory of $(\mathbb{R}, \exp)$ over that of $(\mathbb{R}, \exp|_{[0,1]})$ (the theory of restricted real exponentiation).

In Section 2, we recall some notions and summarize some results from the previous papers [KS] and [K–K]. We will define a logarithmic cross-section of an ordered field $(K, <)$ with natural valuation v and value group $G = vK$ to be an embedding h of G into an additive group complement to the valuation ring, and we will call it strong if it satisfies $vh(g) > g$ for all $g \in G, g < 0$. Suppose that an exponential field (K, f) is a model of restricted real exponentiation, that is, $(K, f|_{[0,1]})$ has the same elementary properties as $(\mathbb{R}, \exp|_{[0,1]})$. Then Theorem 2 states that (K, f) is a model of real exponentiation, that is, it has the same elementary properties as $(\mathbb{R}, \exp)$, if and only if f induces a strong logarithmic cross-section. From this, one obtains Corollary 3 which says that a model of restricted real exponentiation can be turned into a model of real exponentiation if and only if it admits a strong logarithmic cross-section which is surjective (i.e. $h(G)$ is an additive group complement to the valuation ring).

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In Section 3, we discuss the problem of constructing a power series field that is a model of real exponentiation. The search for an exponential power series field is interesting because it could provide a good notion of "maximal exponential field", possibly also giving an idea for a definition of "henselian exponential field". Moreover, an exponential power series field would give us the possibility of applying the strong properties of spherically complete fields in the investigation of exponential terms.

It follows from earlier work done in [KS] that an exponential power series field with real coefficients exists if and only if a totally ordered set Γ exists such that for the Hahn product $\mathbb{R}^\Gamma$, the ordered set $(\mathbb{R}^\Gamma)^{<0}$ is isomorphic to Γ. We construct a Γ which is very close to satisfying the required condition (Theorem 7).

We end Section 3 by showing that, for a large class of groups G, the power series field $\mathbb{R}((G))$ admits a strong logarithmic cross-section. In combination with the logarithm defined on the valuation ring by the logarithmic power series, it gives rise to a (non-surjective) logarithm defined on all positive elements of $\mathbb{R}((G))$.

As an application, we shall give in Section 4 a construction of non archimedean models of real exponentiation. Similar constructions were already given by Dahn and Göring, and later modified in [D–M–M2]. Our construction is different in so far as it exploits the existence of the (non surjective) logarithm at the first step, and requires only one limit process in order to make the logarithm surjective.

2 Preliminaries on left logarithms

Let G be a totally ordered abelian group. Recall that the set of archimedean classes of all nonzero elements of G is endowed with a total ordering given by the rule that $[a] < [b]$ if $|a| \gg |b|$. The chain thus obtained is the **rank** of G. The natural valuation v_G on G is the surjective map which associates to every element $a \neq 0$ its archimedean class $[a]$. Thus, the rank of G will be denoted by $v_G G$. Similarly, given a totally ordered field K, we consider the natural valuation v on its additive group $(K, +, 0, <)$. In this case, the rank carries an extra structure: it forms a totally ordered abelian group G (denoted by vK) if endowed with the addition $[a] + [b] := [ab]$. The natural valuation is now a field valuation, with value group vK. Throughout this paper, the natural valuations of the appearing fields will be denoted by v, and the natural valuations of their value groups will be denoted by v_G. Like field valuations, group valuations satisfy $v(-g) = v(g)$ and the triangle inequality. For more information on natural valuations, see [KS]. Here, let us mention only the following fact. If $g_1, g_2 \in G^{<0} = \{g \in G \mid g < 0\}$, then $v_G g_1 < v_G g_2$ says that $|g_1| \gg |g_2|$, hence it implies that $g_1 < g_2$ (analogously, the natural valuation of an ordered field acts on its negative elements). The residue field of (K, v) will be denoted by $\overline{K}$. For the rest of the paper, K will denote a real closed field with natural valuation v, valuation ring $\mathcal{O}$ and value group G. For a group G, the symbol $G^{<0}$ will denote the totally ordered set of negative elements of the group. Further, $\mathcal{U}^{>0} := \{a \in K \mid va = 0 \wedge a > 0\}$ denotes the set of positive units in K. It is a convex subgroup of $(K^{>0}, \cdot, 1, <)$. Recall that we have the following representations for the ordered groups $(K, +, 0, <)$ and $(K^{>0}, \cdot, 1, <)$ (see [KS], Lemma 3.4 and Theorem 3.8). The former admits a representation as a lexicographic product

$$(K,+,0,<) \simeq \mathbf{A} \amalg (\mathcal{O},+,0,<) \tag{1}$$

where **A** is an arbitrary group complement of $\mathcal{O}$ in $(K,+)$. Endowed with the restriction of the ordering, it is unique up to isomorphism. The archimedean components of **A** are all isomorphic to the ordered additive group of $\overline{K}$, and its rank is the ordered set $G^{<0}$.

An analogous representation of $(K^{>0},\cdot,1,<)$ can be given:

$$(K^{>0},\cdot,1,<) \simeq \mathbf{B} \amalg (\mathcal{U}^{>0},\cdot,1,<) \tag{2}$$

where **B** is an arbitrary group complement of $\mathcal{U}^{>0}$ in $(K,\cdot)$. Endowed with the restriction of the ordering, it is again unique up to isomorphism. Moreover, there is something special about **B**: For $a,b \in K^{>0}$, we have $a \geq b \Rightarrow va \leq vb$. Hence, the map

$$(K^{>0},\cdot,1,<) \;\rightarrow\; (G,+,0,<)\,, \qquad a \mapsto -va = va^{-1} \tag{3}$$

is a surjective group homomorphism preserving $\leq$, with kernel $\mathcal{U}^{>0}$. We find that every complement **B** is isomorphic to $(G,+,0,<)$ through the map $-v$.

Recall that an **exponential** f on K is an isomorphism of ordered groups

$$f:\;(K,+,0,<) \;\rightarrow\; (K^{>0},\cdot,1,<)\,, \tag{4}$$

s.t. $f(\mathcal{O}) = \mathcal{U}^{>0}$. The inverse of an exponential is a **logarithm**. We see that f decomposes into an isomorphism f_r of the ordered additive group $(\mathcal{O},+,0,<)$ onto the ordered multiplicative group $(\mathcal{U}^{>0},\cdot,1,<)$ on the one hand, and an isomorphism

$$f_l:\;\mathbf{A} \;\rightarrow\; \mathbf{B}$$

on the other hand. Such an isomorphism f_r is called a **right exponential**, whereas an isomorphism f_l is a **left exponential**. Conversely, in view of (1) a right and a left exponential can be put together to obtain an exponential of K. (The indeces "l" and "r" refer to the left hand summand resp. the right hand summand of the lexicographic products (1) and (2)). The inverse f_l^{-1} of a left exponential will be called a **(surjective) left logarithm**.

Through the isomorphism $-v$, every isomorphism

$$h:\;(G,+,0,<) \;\rightarrow\; \mathbf{A}$$

gives rise to a surjective left logarithm $h \circ -v$. Conversely, given a surjective left logarithm f_l^{-1}, the map $f_l^{-1} \circ (-v)^{-1}$ is such an isomorphism h. That is, there is a one to one correspondence between surjective left logarithms and isomorphisms of G onto **A**.

Since we are interested in models of real exponentiation, the constructed exponentials have to satisfy certain conditions. We employ a theorem of J.-P. Ressayre [R], which can be stated as follows:

Theorem 1 (J.-P. Ressayre)
Let $(K,<)$ be a real closed ordered field and let $f:\;(K,+,0,<) \simeq (K^{>0},\cdot,1,<)$. If (K,f) is a model of restricted real exponentiation and if f satisfies the axiom scheme

$$x > n^2 \implies f(x) > x^n \qquad (n \in \mathbb{N})\,, \tag{5}$$

then (K,f) is a model of real exponentiation.

This result still holds if one adds restricted analytic functions to (K, f) and $(\mathbb{R}, \exp)$, cf. [D–M–M1], (4.10).

Because of the condition "$x > n^2$", axiom scheme (5) is void for infinitesimals. That is, it gives information only in the case of $vx \leq 0$. It holds in the case $vx = 0$ if the exponential $\overline{f}$ induced by f on $\overline{K}$ satisfies (5) in the place of f (e.g. if $\overline{f}$ is the usual exp on $\overline{K} = \mathbb{R}$), the proof is simple, see e.g. [K-K], Lemma 2.10.

Now we have to consider the case of $vx < 0$. In this case, "$x > n^2$" holds for all $n \in \mathbb{N}$ if only x is positive. Restricted to $K \setminus \mathcal{O}$, axiom scheme (5) is thus equivalent to the assertion

$$vx < 0 \wedge x > 0 \Longrightarrow \forall n \in \mathbb{N} : \; f(x) > x^n \,. \tag{6}$$

But "$\forall n \in \mathbb{N} : f(x) > x^n$" means that $f(x)$ is infinitely bigger than x in the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$. Through the isomorphism f^{-1}, this is equivalent to x being infinitely bigger than $f^{-1}(x)$ in $(K, +, 0 <)$, or in other words, $vf^{-1}(x) > vx$. Hence, (6) is equivalent to

$$vx < 0 \wedge x > 0 \Longrightarrow vf^{-1}(x) > vx \,. \tag{7}$$

(Observe that the condition "$vx < 0 \wedge x > 0$" implies that $f^{-1}(x)$ exists and that $vf^{-1}(x) < 0$.)

Now every $x \in K^{>0}$ can be written as $x = b \cdot c$ where $b \in \mathbf{B}$ and $c \in \mathcal{U}^{>0}$, and $vx = vb$. Then $vf^{-1}(x) = v(f^{-1}(b) + f^{-1}(c)) = vf^{-1}(b)$ since $c \in \mathcal{U}^{>0}$ implies that $vf^{-1}(c) \geq 0 > vf^{-1}(b)$. So (7) holds if and only if it holds for f_l in the place of f. Hence, (7) is equivalent to

$$x \in \mathbf{B} \wedge x > 0 \Longrightarrow vf_l^{-1}(x) > vx \,. \tag{8}$$

With $g = vx$ and the isomorphism $h = f_l^{-1} \circ (-v)^{-1} : G \to \mathbf{A}$, and in view of $(-v)^{-1}(vx) = x^{-1}$ and $vf_l^{-1}(x^{-1}) = v(-f_l^{-1}(x)) = vf_l^{-1}(x)$, condition (8) translates to

$$vh(g) > g \quad \text{for all } g \in G^{<0} \,. \tag{9}$$

In view of Ressayre's Theorem, we have proved the following:

Theorem 2 *Let f be an exponential on $(K, <)$ such that $\overline{f}$ satisfies (5) on $\overline{K}$, and such that (K, f) is a model of restricted real exponentiation. Then (K, f) is a model of real exponentiation if and only if (8) holds, or equivalently, if and only if (9) holds.*

Here again, the result still holds if one adds restricted analytic functions.

Remark The following holds:
If on a nonarchimedean ordered field $(K, <)$, an isomorphism (4) satisfies $f(a) > a$ for all infinitely big elements $a \in K^{>0}$, then it satisfies $f(a) > a^n$ for all infinitely big elements $a \in K^{>0}$ and all $n \in \mathbb{N}$.

Indeed, $f(a) > a$ implies $vf(a) \leq va$. Suppose that $vf(a) = va$ for some infinitely big $a \in K$ (note that $va < 0$). By assumption, $f(a/2) > a/2$, which yields that $vf(a/2) \leq v(a/2)$. But then, $va = vf(a) = vf(a/2)^2 = 2vf(a/2) \leq 2v(a/2) = 2va < va$, a contradiction. Hence, $vf(a) < va$ and thus $f(a) > a^n$ for all infinitely big elements $a \in K^{>0}$ and all $n \in \mathbb{N}$.

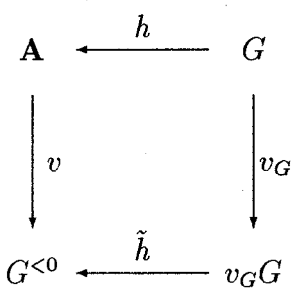
In view of the above theorem, it is natural to ask for the existence of isomorphisms h satisfying equation (9). Not every real closed field will admit such an isomorphism. For instance, if G is a countable divisible ordered agelian group, then the real closed power series field $\mathbb{R}((G))$ does not admit such an isomorphism, since every additive group complement to the valuation ring is uncountable (having $\mathbb{R}$ as its components, cf. our remarks subsequent to Theorem 6 below).

So, we will rather start by asking for an *embedding* h of the value group G into an additive complement to the valuation ring. Such an embedding will be called a **logarithmic cross-section**. If in addition it satisfies condition (9), then we call it a **strong logarithmic cross-section**. Every logarithmic cross-section h gives rise to a *not necessarily surjective* left logarithm $f_l^{-1} = h \circ -v$, and vice versa. Then h is surjective if and only if f_l^{-1} is. Following the terminoloy introduced in [K-K], a left logarithm f_l^{-1} (respectively, left exponential f_l) satisfying (8) will be called a **strong left logarithm** (respectively, **strong left exponential**). Hence, h is strong if and only if f_l^{-1} is.

If a real closed field K admits a surjective strong logarithmic cross-section, then it admits a strong left exponential f_l. If it also admits some exponential with which it is a model of restricted real exponentiation, then we let f_r be the right part of this exponential. Note that $f|_{[0,1]} = (f_r)|_{[0,1]}$ for every exponential f since $[0,1] \subset \mathcal{O}$. So we can put f_l and f_r together to obtain an exponential f such that (K, f) is a model of restricted real exponentiation. By Theorem 2, (K, f) is then a model of real exponentiation. We have proved:

Corollary 3 *If an exponential field is a model of restricted real exponentiation and admits a surjective strong logarithmic cross-section, then it admits an exponential with which it is a model of real exponentiation.*

Now recall that every embedding (resp. isomorphism) of ordered abelian groups induces canonically an embedding (resp. isomorphism) of their ranks as ordered sets (c.f. [KS]). In particular, such an embedding h induces an embedding $\tilde{h}$ such that the following diagram commutes, i.e. h is a **lifting** of $\tilde{h}$:



and we have

Lemma 4 *For every* $g \in G^{<0}$,

$$\tilde{h}(v_G g) > g \iff vh(g) > g \; .$$

That is, h is a strong logarithmic cross-section if and only if

$$\tilde{h}(v_G g) > g \quad \text{for all } g \in G^{<0} . \tag{10}$$

If h is an isomorphism, then so is $\tilde{h}$ (in this case, it is just the inverse of a "group exponential" as defined in [KS]).

Note that every ordered abelian group admits an embedding $s : v_G G \to G^{<0}$ of ordered sets such that $v_G \circ s$ is the identity on $v_G G$ (for $\alpha \in v_G G$, we just have to set $s\alpha = g$ where $g \in G^{<0}$ is an arbitrary element of value $v_G g = \alpha$). We will call such a map a **group cross-section** of the valued group (G, v_G). It can be used to get

Lemma 5 *Let G be any ordered abelian group such that $v_G G$ admits an automorphism σ satisfying $\sigma\alpha > \alpha$ for all $\alpha \in v_G G$. Then for every group cross-section s of G, the embedding $\tilde{h} := s \circ \sigma : v_G G \to G^{<0}$ will satisfy condition (10).*

Indeed, $v_G \tilde{h}(v_G g) = \sigma v_G g > v_G g$ and thus a fortiori $\tilde{h}(v_G g) > g$ if $g \in G^{<0}$. Note that there are plenty of groups satisfying the hypothesis of the lemma. For instance, this is the case if $v_G G$ is isomorphic to an arbitrary nontrivial ordered abelian group, as an ordered set.

Now the question arises whether an embedding (resp. isomorphism) $\tilde{h}$ can be lifted to an embedding (resp. isomorphism) h. (Cf. the related notion of "lifting property" as used in [K–K].) Such a lifting always exists if $\mathbf{A}$ is rich enough, i.e. if it is a Hahn product. This in turn is the case if the field K is a power series field. We will discuss this special case in the next section.

3 The case of power series fields over the reals

If G is an arbitrary ordered abelian group, then the power series field $K := \mathbb{R}((G))$ is a formally real field, and it is real closed if and only if G is divisible (which we shall always assume here). Further, K carries a canonical valuation v which associates to every formal power series the minimum of its support. It also carries a natural ordering $<$ such that v is the natural valuation of the ordered field $(K, <)$. The residue field of (K, v) is $\mathbb{R}$, and its value group is G. The valuation ring $\mathcal{O}$ of (K, v) is the power series ring $\mathbb{R}[[G]]$ which consists of all formal power series whose support is a subset of $G^{\geq 0} = \{g \in G \mid g \geq 0\}$. On $\mathcal{O}$, the exponential and all other analytic functions can be defined by means of their associated power series. Then, K together with the restrictions of these functions to the interval $[0, 1]$ will have the same elementary properties as $\mathbb{R}$ together with the corresponding restricted functions (cf. [D–M–M1], Corollary 2.11). Let us denote the exponential defined on $\mathcal{O}$ by f_r. The problem is whether f_r can be extended to an exponential f on the whole field in such a way that (K, f) is a model of real exponentiation, or in other words, whether there exists a strong left exponential (or equivalently, a strong surjective left logarithm). As we have seen in the last section, the answer is negative in general. Even for the existence of a mere left exponential, a very restrictive necessary and sufficient condition on the value group is imposed. Let us derive this condition from the results of the last section. In the following, let $\mathbb{R}^\Gamma$ denote the Hahn product with coefficients in $\mathbb{R}$ and exponents in the totally ordered set Γ. In the case of the power series field K, we can take the additive group complement $\mathbf{A}$ of the valuation ring to be the ordered ring $\{a \in \mathbb{R}((G)) \mid \text{support}(a) \subset G^{<0}\}$, which is canonically isomorphic to $\mathbb{R}^{G^{<0}}$. We have shown in the last section that K admits a left exponential f_l if and only if there exists a surjective logarithmic cross-section h, and that f_l is strong if and only if h is.

Theorem 6 *The field $\mathbb{R}((G))$ admits an exponential if and only if there is an isomorphism*

$$h: \, G \simeq \mathbb{R}^{G^{<0}} . \tag{11}$$

of ordered groups (which in view of $\mathbf{A} = \mathbb{R}^{G^{<0}}$ is a surjective logarithmic cross-section). If in addition h is strong, that is, satisfies condition 9, then $\mathbb{R}((G))$ admits a strong left exponential and thus also an exponential with which it is a model of real exponentiation.

(Cf. [K-K], Corollary 2.22.) As it is the case for Ressayre's Theorem, the latter result still holds if one adds restricted analytic functions.

Let us write $\Gamma := v_G G$. Condition (11) of the theorem implies the existence of an embedding $\tilde{h} : \Gamma \to G^{<0}$. But it also implies that G is isomorphic to $\mathbb{R}^{\Gamma}$. Consequently, condition (11) implies the existence of an isomorphism

$$\tilde{h} : \Gamma \simeq (\mathbb{R}^{\Gamma})^{<0}$$

of ordered sets. Conversely, if such an ordered set Γ is given, then $G := \mathbb{R}^{\Gamma}$ satisfies $\Gamma \simeq G^{<0}$ and thus $G = \mathbb{R}^{\Gamma} \simeq \mathbb{R}^{G^{<0}}$. Under the isomorphism $G \simeq \mathbb{R}^{\Gamma}$, the group valuation v_G is equal to the canonical minimum support valuation of $\mathbb{R}^{\Gamma}$ (which is defined as above for power series fields). Hence by virtue of Lemma 4, h is strong if and only if $\tilde{h}$ satisfies that $\tilde{h}(\min \text{support}(g)) > g$ for every $g \in (\mathbb{R}^{\Gamma})^{<0}$.

The existence of such a totally ordered set Γ is unknown. However, we have a construction that is amusingly near to the wanted result:

Theorem 7 *There is an ordered set Γ such that $(\mathbb{R}^{\Gamma})^{\leq 0}$ is order isomorphic to Γ.*

To construct Γ, we set $\Gamma_1 := \mathbb{R}^{\leq 0}$ and $\Gamma_{n+1} := (\mathbb{R}^{\Gamma_n})^{\leq 0}$ for $n \in \mathbb{N}$. Since $\mathbb{R}^{\leq 0}$ has a maximal element, $\mathbb{R}$ can be viewed as a convex subgroup of $\mathbb{R}^{\Gamma_1}$. By induction, it is shown that $\mathbb{R}^{\Gamma_{n-1}}$ is a convex subgroup of $\mathbb{R}^{\Gamma_n}$. This proves that Γ_n is a final segment of Γ_{n+1}. Now we let $\Gamma := \bigcup_{n \in \mathbb{N}} \Gamma_n$. Since every Γ_n is a final segment of Γ, every wellordered subset S of Γ is already contained in some Γ_n (just take n such that the first element of S lies in Γ_n). Hence, an element of $(\mathbb{R}^{\Gamma})^{\leq 0}$ with support S is actually an element of $\Gamma_{n+1} = (\mathbb{R}^{\Gamma_n})^{\leq 0}$. This fact gives rise to an order isomorphism of $(\mathbb{R}^{\Gamma})^{\leq 0}$ onto Γ. Its inverse $\tilde{h}$ is an order isomorphism $\Gamma \simeq (\mathbb{R}^{\Gamma})^{\leq 0}$ which satisfies $\tilde{h}(\min \text{support}(g)) > g$ for every $0 \neq g \in \mathbb{R}^{\Gamma}$. This is seen as follows. In view of our identification of $\{g \in (\mathbb{R}^{\Gamma})^{\leq 0} \mid \text{support}(g) \subset \Gamma_n\}$ with Γ_{n+1}, we can consider $\tilde{h}$ to be the identity. Now if $\min \text{support}(g) \in \Gamma_n \backslash \Gamma_{n-1}$, then $g \in \Gamma_{n+1} \backslash \Gamma_n$. Since Γ_n is a final segment of Γ_{n+1}, it follows that $\min \text{support}(g) > g$.

If Δ denotes the subset of Γ which contains all but the maximal element of Γ, then $\tilde{h}$ is an order isomorphism from Δ onto $(\mathbb{R}^{\Gamma})^{<0}$. This isomorphism induces an order embedding of $(\mathbb{R}^{\Delta})^{<0}$ into Δ. Conversely, there is an order embedding of Δ into $(\mathbb{R}^{\Delta})^{<0}$. However, we do not know whether both sets are order isomorphic.

Remark To find $\Gamma \simeq (\mathbb{R}^\Gamma)^{<0}$, it is actually sufficient to construct Γ such that $\mathbb{R}^\Gamma \simeq \Gamma$. Indeed, Γ will then be an ordered group and $\mathbb{R}^\Gamma$ will be an ordered field. But it can be shown that for every ordered field K, we have that $K^{<0} \simeq K$ as ordered sets. Thus also $(\mathbb{R}^\Gamma)^{<0} \simeq \Gamma$ as required.

Since we do not know any $\Gamma \simeq (\mathbb{R}^\Gamma)^{<0}$, we do not know a group G with property (11). In fact, we can construct groups G which admit isomorphisms $\tilde{h} : v_G G \to G^{<0}$ that satisfy condition (10) (cf. [K–K] and [KF1]), but the construction method is not capable of producing groups which are Hahn products. Consequently, we do not know whether there exist exponential power series fields. Let us summarize what remains open:

Open Problems *Do there exist exponential power series fields? Do there exist exponential power series fields which are models of real exponentiation? Equivalently, do there exist ordered sets* Γ *such that* $\Gamma \simeq (\mathbb{R}^\Gamma)^{<0}$. *If so, can an isomorphism* $\tilde{h}$ *be constructed which satisfies* $\tilde{h}(\min \mathrm{support}(g)) > g$ *for every* $g \in (\mathbb{R}^\Gamma)^{<0}$*?*

However, we can prove:

Theorem 8 *Assume that the rank* $v_G G$ *of the ordered abelian group* G *admits an automorphism* σ *satisfying* $\sigma\alpha > \alpha$ *for all* $\alpha \in v_G G$. *Then the power series field* $\mathbb{R}((G))$ *admits a (not necessarily surjective) strong logarithmic cross-section.*

Proof: According to Lemma 5, we can choose an embedding $\tilde{h} : v_G G \to G^{<0}$ which satisfies condition (10). Note that $\mathbf{A}$ is archimedean-complete (that is, it is maximal and all its components are $\mathbb{R}$). Hence by Hahn's embedding theorem, the embedding $\tilde{h}$ of $v_G G$ into $G^{<0} = v\mathbf{A}$ lifts to an embedding h of G into $\mathbf{A}$. Moreover, since $\tilde{h}(v_G g) > g$, Lemma 4 shows that $vh(g) > g$ for all $g \in G^{<0}$, as required. □

As an application of this last theorem, we shall construct in the next section nonarchimedean models of real exponentiation, which are not power series fields, but countable unions of power series fields. Indeed, a common method to obtain surjectivity of a map is to construct the union over a suitable countably infinite chain of fields. In the following, we will apply such a construction to strong logarithmic cross-sections.

4 Going to the limit

In [D–M–M2], L. van den Dries, A. Macintyre and D. Marker modify the method of Dahn and Göring for the construction of nonarchimedean exponential fields which are models of real exponentiation. We will give an alternative modification. Using the results of the last section, we eliminate one of the two limit processes used in [D–M–M2].

- **Construction of the left exponential.**

To get started, let G_0 and $K_0 = \mathbb{R}((G_0))$ be as in Theorem 8. Let $\mathbf{A}_0$ be a group complement of $\mathbb{R}[[G_0]]$ in K_0 and $h_0 : G_0 \to \mathbf{A}_0$ a logarithmic cross-section of K_0. Now assume that we have already constructed G_{n-1}, K_{n-1}, $\mathbf{A}_{n-1}$ and the logarithmic cross-section

$$h_{n-1} : G_{n-1} \to \mathbf{A}_{n-1}$$

i.e. satisfying that

$$vh_{n-1}(g) > g \quad \text{for all } g \in G_{n-1}^{<0} . \tag{12}$$

Since G_{n-1} is isomorphic to a subgroup of $\mathbf{A}_{n-1}$ through h_{n-1}, we can take G_n to be a group containing G_{n-1} as a subgroup and admitting an isomorphism h_n onto $\mathbf{A}_{n-1}$ which extends h_{n-1} . We set $K_n := \mathbb{R}((G_n))$. Hence, $K_{n-1} \subset K_n$ canonically (the elements of K_{n-1} being those elements of K_n whose support is a subset of G_{n-1}). Further, we choose a group complement $\mathbf{A}_n$ for the valuation ring $\mathbb{R}[[G_n]]$ such that $\mathbf{A}_n$ contains $\mathbf{A}_{n-1}$. In this way, h_n appears as an embedding of G_n into $\mathbf{A}_n$ which extends h_{n-1} . We show that h_n is again a logarithmic cross-section. For $g \in G_n$, the image $h_n(g)$ lies in $\mathbf{A}_{n-1}$, and $vh_n(g)$ lies in its value set $G_{n-1}^{<0}$. Consequently, in (12) we may replace $g \in G_{n-1}^{<0}$ by $vh_n(g)$ for $g \in G_n^{<0}$. But $vh_{n-1}(vh_n(g)) > vh_n(g)$ implies $h_{n-1}(vh_n(g)) > h_n(g)$, because $h_n(g) < 0$ and $h_{n-1}(vh_n(g)) < 0$. Since h_n extends h_{n-1} , this may be read as $h_n(vh_n(g)) > h_n(g)$. Since h_n is order preserving, this in turn implies $vh_n(g) > g$. Thus, we have proved that (12) holds with n in the place of $n-1$.

By our induction on n, we obtain a chain of fields K_n , $n \in \mathbb{N}$. Now we take $K_\omega := \bigcup_{n\in\mathbb{N}} K_n$, $f_\omega := \bigcup_{n\in\mathbb{N}} f_n$ and $h_\omega := \bigcup_{n\in\mathbb{N}} h_n$. Also the groups G_n form a chain, and their union $G_\omega := \bigcup_{n\in\mathbb{N}} G_n$ is the value group of K_ω . Similarly, the group complements $\mathbf{A}_n$ form a chain, and their union $\mathbf{A}_\omega := \bigcup_{n\in\mathbb{N}} \mathbf{A}_n$ is a group complement for the valuation ring $\mathbb{R}[[G_\omega]]$ in K_ω . By construction, we have $\mathbf{A}_{n-1} = h_n(G_n)$ for all n. Consequently, $h_\omega : G_\omega \to \mathbf{A}_\omega$ is surjective. Moreover, h_ω satisfies property (9). It follows that the surjective map $f_{l,\omega} := (h_\omega \circ -v)^{-1}$ is a left exponential satisfying condition (8).

- **Construction of the right exponential.**

Let $n \in \mathbb{N}$ and a be an element of the valuation ring $\mathbb{R}[[G_n]]$ of K_n . Then we can write $a = r + \varepsilon$ with $r \in \mathbb{R}$ and $v\varepsilon > 0$. We set

$$f_{r,n}(a) := \exp(r) \cdot \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \, ;$$

note that the second factor is again an element of $\mathbb{R}[[G_n]]$. This definition yields that $(K_n , f_{r,n})$ is a model of restricted real exponentiation; this follows from Corollary (2.11) of [D–M–M1]. Further, $f_{r,n+1}$ is immediately seen to be an extension of $f_{r,n}$, and since also $(K_{n+1} , f_{r,n+1})$ is a model of restricted real exponentiation, it follows from Wilkie's theorem on the model completeness of the restricted exponential function (cf. [W1]) that

$$(K_n , f_{r,n}|_{[0,1]}) \subset (K_{n+1} , f_{r,n+1}|_{[0,1]})$$

is an elementary extension. Setting $f_{r,\omega} := \bigcup_{n\in\mathbb{N}} f_{r,n}$, we obtain that $(K_\omega , f_{r,\omega})$ is the union over an elementary chain and thus is itself a model of restricted real exponentiation. Moreover, our definition of the $f_{r,n}$ yields that $f_{r,\omega}$ coincides with exp on the subfield $\mathbb{R}$ of K_ω .

Now we let f_ω be the exponential on K_ω which is induced by $f_{l,\omega}$ and $f_{r,\omega}$. Then f_ω satisfies (5), and in view of $f_{r,\omega}|_{[0,1]} = f_\omega|_{[0,1]}$, we see that (K_ω , f_ω) is a model of restricted real exponentiation. From Theorem 1 we conclude that (K_ω , f_ω) is a model of real exponentiation. This completes our construction.

Our construction actually gives a little bit more. If (K, f) is a model of real exponentiation, then it admits a strong logarithmic cross-section h. If we set $G_0 := vK$ and embed K in the power series field $K_0 := \mathbb{R}((G))$, then h will still be a strong logarithmic cross-section of K_0. Now our construction will yield an extension (K_ω, f_ω) of (K, f). By Wilkie's Theorem on the model completeness of $(\mathbb{R}, \exp)$, [W2], this embedding is elementary. We have proved:

Corollary 9 *Every model (K, f) of real exponentiation can be elementarily embedded in a model (K_ω, f_ω) which is a countable union of power series fields.*

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