

MODEL THEORY – EXERCISE 2

To be submitted on Wednesday 27.04.2011 by 14:00 in the mailbox.

Definition.

Suppose L is a signature, $L' \subseteq L$, M, N are L -structures.

- (1) We say that M is a *substructure* of N , denoted by $M \subseteq N$ if the universe of M is a subset of the universe of N and $\text{id} : M \rightarrow N$ (the identity map) is an embedding.
- (2) We say that M is the L' -*reduct* of N , denoted by $M = N \upharpoonright L'$ if M is the L' -structure whose universe is the same universe as N and all symbols in L' are interpreted as in N .
- (3) Given a formula φ , we write $\varphi(x_0, x_1, \dots, x_{n-1})$ to indicate that the set of free variables of φ is contained in $\{x_0, \dots, x_{n-1}\}$.
- (4) We call a formula φ a *sentence* if it has no free variables.
- (5) We call a formula φ *quantifier free* if it does not contain the quantifiers \exists, \forall .
- (6) We call a formula φ *universal* if it is of the form $\forall x_0 \forall x_1 \dots \forall x_{n-1} \psi$ where ψ is quantifier free.
- (7) We call a formula φ *existential* if it is of the form $\exists x_0 \exists x_1 \dots \exists x_{n-1} \psi$ where ψ is quantifier free.
- (8) We call a formula φ *equational* if it is of the form $\forall x_0 \dots \forall x_{n-1} \psi$ where ψ is atomic.
- (9) For 2 L -sentences φ and ψ , we write $\psi \models \varphi$ if for every L -structure M , $M \models \psi \Rightarrow M \models \varphi$.
- (10) We say that two L -sentences φ and ψ are *elementarily equivalent* if for every L -structure M , $M \models \varphi$ iff $M \models \psi$.

Question 1.

In the following clauses you are given a signature and a mathematical statement in English. Write this statement as a sentence in the signature.

- (1) $L = \{f, +, -, |, 0, <\}$ where $f, |$ are unary function symbols, $+, -$ are binary function symbols, 0 is a constant and $<$ is a binary relation symbol. The function f is a continuous function when we think of it as a function from \mathbb{R} to \mathbb{R} , and interpret all symbols as in \mathbb{R} .
- (2) $L = \{f, <, 0\}$ where $f, <, 0$ are as in (1). The function f is a continuous function in 0 (when we think of it as a function from \mathbb{R} to \mathbb{R}).
- (3) $L = \{<, c_0, c_1\}$ where $<$ is a binary relation symbol, c_0, c_1 are constants. $<$ is a dense linear order (between every 2 elements there is a third one), and c_0 is the first element, c_1 is the last element.
- (4) $L = \{S\}$ where S is a unary relation symbol. The universe contains exactly 5 elements, exactly 3 of them are in S .
- (5) $L = \{R\}$ where R is a binary relation symbol. R is a graph of a function.
- (6) $L = \{R\}$ as in (5). R is a graph of a surjective function.

- (7) $L = \{+, \cdot, 0, 1\}$ where $+, \cdot$ are binary function symbols, $0, 1$ are constants (this is the signature of rings). For every polynomial of degree 5 there exists a root.

Question 2.

Suppose $M \subseteq N$ are 2 L-structures (so M is a substructure of N).

- (1) Suppose $\varphi(x_0, \dots, x_{n-1})$ is a quantifier free formula, and $a_0, \dots, a_{n-1} \in M$. Show that $M \models \varphi[a_0, \dots, a_{n-1}]$ iff $N \models \varphi[a_0, \dots, a_{n-1}]$.
- (2) Suppose $\varphi(x_0, \dots, x_{n-1})$ is a universal formula (so of the form $\forall y_0 \dots \forall y_{m-1} \psi(y_0, \dots, y_{m-1}, x_0, \dots, x_{n-1})$). Show that if $N \models \varphi[a_0, \dots, a_{n-1}]$ then $M \models \varphi[a_0, \dots, a_{n-1}]$.
- (3) Suppose $\varphi(x_0, \dots, x_{n-1})$ is an existential formula. Show that if $M \models \varphi[a_0, \dots, a_{n-1}]$ then $N \models \varphi[a_0, \dots, a_{n-1}]$.
- (4) Let $L = \{P\}$ where P is a unary relation symbol. Show that the sentence $\forall x P(x)$ is not elementarily equivalent to an existential sentence.

Question 3.

- (1) Suppose $f : A \rightarrow B$ is a homomorphism between two L-structures. Show that the image of f is a substructure of B .
- (2) Let $\{A_i \mid i \in I\}$ be structures for a signature L , and let $L' \subseteq L$. Show that $(\prod_i A_i) \upharpoonright L' = \prod_i (A_i \upharpoonright L')$.
- (3) Show that if φ is equational then φ is preserved under homomorphic images, products, and substructures.

Explanation: you need to show that

- (a) If $A_i \models \varphi$ for all $i \in I$, then $\prod A_i \models \varphi$.
- (b) If $A \subseteq B$ and $B \models \varphi$ then $A \models \varphi$.
- (c) If $f : A \rightarrow B$ is a surjective homomorphism and $A \models \varphi$ then $B \models \varphi$.