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MODEL THEORY – EXERCISE 1

To be submitted on Wednesday 20.04.2011 by 14:00 in class.

Definition.

Suppose L is a language (signature), M, N are L-structures.

(1) A *homomorphism* $f : M \to N$ is a function such that
  (a) For any n-ary relation symbol R,
  $(a_1, \ldots, a_n) \in R^M \Rightarrow (f(a_1), \ldots, f(a_n)) \in R^N$.
  (b) For any n-ary function symbol F,
  $F^M(a_1, \ldots, a_n) = b \Rightarrow F^N(f(a_1), \ldots, f(a_n)) = f(b)$.
  (c) For any constant c, $F(c^M) = c^N$.

(2) An *embedding* $f : M \to N$ is a homomorphism $f : M \to N$ such that in (a) above, $\Rightarrow$ is replaced by $\Leftrightarrow$.

(3) A homomorphism is called an *isomorphism* if it is an embedding and it is onto.

(4) A homomorphism $f : M \to M$ is called an *automorphism* if it is an isomorphism from M onto M.

(5) Denote $M \cong N$ when there exists an isomorphism $f : M \to N$.

(6) A group $(G, +, <)$ is an *ordered abelian group* if $(G, +)$ is an abelian group, $(G, <)$ is a linear ordering and $a < b \Rightarrow a + c < b + c$ for all $a, b, c \in G$.

Question 1.

Let A, B, C be structures for a language L.

(1) Show that embeddings are injective (one to one).

(2) Show that if $f : A \to B$ and $g : B \to C$ are homomorphisms then $g \circ f : A \to C$ is a homomorphism.

(3) Show that $f : A \to B$ is an isomorphism iff f is a homomorphism and there exists a $g : B \to A$ such that $f \circ g = \mathrm{id}_B$, $g \circ f = \mathrm{id}_A$.

(4) Show that $\cong$ is an equivalence relation between L-structures.

Question 2.

(1) Let $L = \{P\}$ where P is a predicate (1-place relation). Find an example of two L-structures A, B such that there exists an injective homomorphism from A onto B, so that they are not isomorphic.
Solution: take $A = B = \{1\}$, $P^A = \emptyset$, $P^B = \{1\}$.

(2) Let L be the language of groups, $L = \{+\}$ where $+$ is a 2-place function symbol (but you may write $a + b$ instead of $+(a, b)$). Let M, N be abelian groups. Show that a group homomorphism $h : M \to N$ is exactly a homomorphism of structures.

(3) Let $L = \{+, <\}$ where $<$ is a binary relation symbol (but you may write $a < b$ instead of $<(a, b)$). Let M, N be ordered abelian groups. Show that if $f : M \to N$ is an injective homomorphism of structures which is onto then f is an isomorphism.

Solution: if $f(a) < f(b)$ it must be that $b \leqslant a$ or $a < b$ but it cannot be that $b \leqslant a$.

Question 3.

Let $L = \{P, R\}$ where R is a binary relation symbol and P is a predicate. Describe all possible L-structures of size 2 upto isomorphism, i.e. give a list of L-structures of size 2 such that any L-structure is isomorphic to exactly one of them. Use the following steps:

(1) Write down all structures to L with universe $\{1, 2\}$.
(2) Divide them into $\cong$ equivalence classes.
(3) Show that every structure is isomorphic to one of these structures.

Solution: $P = \emptyset$ and: $R = \emptyset$, $R = \{(1,1)\}$, $R = \{(1,2)\}$, $R = \{(1,1), (2,2)\}$, $R = \{(1,1), (2,1)\}$, $R = \{(1,2), (2,1)\}$, $R = \{(1,1), (1,2)\}$ $R = \{1,2\}^2 \setminus \{(1,2)\}$, $R = \{1,2\}^2 \setminus (1,1)$, $R = \{1,2\}^2$; $P = \{1\}$ (all possibilities for R – all subsets of $\{1,2\}^2$ because there are no non-trivial isomorphisms from $\{1,2\}$ onto $\{1,2\}$ fixing 1) ; $P = \{1,2\}$ (same possibilities for R as in the case $P = \emptyset$). Totally there are $10 + 16 + 10 = 36$ structures upto isomorphism.

Question 4.

Suppose M is a structure, $A \subseteq M$. We let $\mathrm{Aut}(M/A)$ be the set of all automorphisms of M that fix A, i.e.

$$\{\sigma \in \mathrm{Aut}(M) \,|\forall x \in A \,(\sigma(x) = x)\}.$$

We let $\mathrm{Aut}(M/[A])$ be the set of all automorphisms of M that fix A setwise, i.e.

$$\left\{\sigma \in \mathrm{Aut}(M) \,\middle|\forall x \in A \left(\sigma(x) \in A \,\&\, \sigma^{-1}(x) \in A\right)\right\}$$

(1) Show that $\mathrm{Aut}(M/A)$ is a group with composition $(\circ)$.
(2) Show that $\mathrm{Aut}(M/[A])$ is a group with composition.
(3) Show that $\mathrm{Aut}(M/A)$ is a normal subgroup of $\mathrm{Aut}(M/[A])$.
Solution: if $\sigma \in \mathrm{Aut}(M/A)$, and $h \in \mathrm{Aut}(M/[A])$ then for all $x \in A$, $h\sigma h^{-1}(x) = hh^{-1}(x)$ (because $h^{-1}(x) \in A$ and σ fixes A) so it is x.