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MODEL THEORY – EXERCISE 12

To be submitted on Wednesday 6.07.2011 by 14:00 in the mailbox.

Definition. Let L be a signature.

1. (1) A theory T is said to be *absolutely categorical* if $M, N \models T \Rightarrow M \cong N$.
2. (2) A theory T is said to be λ*-categorical* for some cardinal λ if there exists a model of size λ and $M, N \models T, |M| = |N| = \lambda \Rightarrow M \cong N$.
3. (3) We write $M \prec N$ when M is an elementary substructure of N – if $\varphi(\bar{x})$ is a formula and $\bar{a} \in M$ then $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$.

Question 1.
Show that if $M_1 \subseteq M_2 \subseteq M_3$ are L-structures (M_1 is a substructure of M_2 and M_2 is a substructure of M_3) and $M_2 \prec M_3, M_1 \prec M_3$, then $M_1 \prec M_2$.
Solution: immediate from the definition.

Question 2.
Let σ be a signature.

(1) Assume that σ is finite. Show that if M, N are two finite structures such that $M \equiv N$ then $M \cong N$.
Moreover, show that if M is a finite σ-structure, then there is a sentence φ such that $M \models \varphi$ and if $N \models \varphi$ then $N \cong N$.
Solution: suppose $M = \{a_1, \ldots, a_n\}$. Let $\alpha(x_1, \ldots, x_n)$ say that the x_is are distinct and that the universe is $\{x_1, \ldots, x_n\}$. Let R be a k-ary relation symbol. Let $\psi_R(x_1, \ldots, x_n)$ be

$$\bigwedge_{(a_{i_1},\ldots,a_{i_k})\in R^M} R(x_{i_1}, \ldots, x_{i_n}) \wedge \bigwedge_{(a_{i_1},\ldots,a_{i_k})\notin R^M} \neg R(x_{i_1}, \ldots, x_{i_k}).$$

For a k-ary function symbol F, let $\psi_F(x_1, \ldots, x_n)$ be

$$\bigwedge_{F(a_{i_1},\ldots,a_{i_k})=a_{i_j}} F(x_{i_1}, \ldots, x_{i_n}) = x_{i_j}.$$

Now note that $M \models \exists x_1 \ldots x_n \alpha(\bar{x}) \wedge \bigwedge_{R\in L} \psi_R(\bar{x}) \wedge \bigwedge_{F\in L} \psi_F(\bar{x})$, so this sentence holds in N. Let $\{b_1, \ldots, b_n\} \subseteq N$ witness this. Then $N = \{b_1, \ldots, b_n\}$ (because of α) and the function $a_i \mapsto b_i$ is an isomorphism because of ψ_R and ψ_F.

(2) Now prove (1) (without the "moreover") for arbitrary σ.
Solution: Suppose $M \equiv N$ are finite but $M \ncong N$. Then $|M| = |N|$ but for every injective and surjective function $f : M \to N$, f is not an isomorphism. This means that there is some relation (or function symbol), $R_f \in \sigma$, that witness this, i.e. there are $\bar{a} \in M^k$ such that $\bar{a} \in R_f^M$ but $f(\bar{a}) \notin R_f^N$. The number of such functions is finite (bounded by $|M|^{|M|}$, even less). Let A be the set of all such functions. Now let $\sigma' = \{R_j \,|f \in A\}$. Then σ' is a finite signature. So there is a sentence φ as in (1). Since $\varphi \in Th(M)$, $N \models \varphi$, i.e. $N \restriction \sigma'$ is isomorphic to $M \restriction L'$, and let $f : M \to N$ be an σ' isomorphism.

But then $R_f \in \sigma'$ and this is a contradiction. (Another solution is to do it by induction on $|\sigma|$).

(3) Conclude that a theory T is absolutely categorical iff T is complete and has only finite models.
Solution: suppose T complete and has only finite models. Then if $M, N \models T$ then $M \equiv N$ so use (2). If T is categorical, and M is an infinite model, then by the upwards Lowenheim-Skolem theorem, T has a model of any infinite cardinality, and in particular it has a model which is not isomorphic to M. This means that all models of T are finite. On the other hand, if $M, N \models T$ then $M \equiv N$ so T is complete.

Question 3.
Let $L = \{<\}$ where $<$ is a binary relation symbol. Let DLO (in class it was denoted by $DLOWEP$) be the theory of densely ordered (between any two points there is another point) linear orders with no end points (i.e. there is no minimal or maximal element).

(1) Write down the axioms of DLO.
(2) Prove that DLO is $\aleph_0$-categorical.
Hints: assume that $M, N \models DLO$.
  (a) Suppose $f : A \to B$ is a map such that $|A| = |B|$ is finite, $A \subseteq M, B \subseteq N$ and f is an isomorphism (i.e. order preserving). Suppose $a \in M$, show that there is some $f' \supseteq f$ (i.e. extending f) such that $a \in Dom(f')$.
  Solution: if $a \in Dom(f)$ let $f' = f$. Otherwise, if $a > Dom(f)$, find some $b > Im(f)$ (why exists?) and let $f' = f \cup \{(a, b)\}$. If $a < Dom(f)$, then find $b < Im(f)$ and let $f' = f \cup \{(a, b)\}$. Otherwise, let $a_1 < a < a_2$ be such that $a_1, a_2 \in Dom(f)$ and there is no $a' \in Dom(f)$ such that $a_1 < a' < a_2$. Then let b be between $f(a_1)$ and $f(a_2)$ and let $f' = f \cup \{(a, b)\}$.
  (b) Suppose $f : A \to B$ is a map such that $|A| = |B|$ is finite, $A \subseteq M, B \subseteq N$ and f is an embedding (i.e. order preserving). Suppose $b \in N$, show that there is some $f' \supseteq f$ (i.e. extending f) such that $b \in Im(f')$.
  Solution: do the same as before, or consider f^{-1}.
  (c) Now, assume $|M| = |N| = \aleph_0$ and let $M = \{a_i \mid i < \omega\}, N = \{b_i \mid i < \omega\}$. Define a sequence of functions f_i such that
    - $Dom(f_i), Im(f_i)$ are finite.
    - $f_i : Dom(f_i) \to Im(f_i)$ is an isomorphism.
    - $a_i \in Dom(f_{2i+1}), b_i \in Im(f_{2i+2})$.
    - $f_i \subseteq f_{i+1}$.
  (d) Finish the proof.
  Solution: take $f = \bigcup f_i$.
(3) Deduce that DLO is complete.
Solution: Immediate by the Los-Vaught test.
(4) Prove that DLO has quantifier elimination (hint: use Exercise 4).
Solution: in Ex. 4, Q 3, it was shown that $Th(\mathbb{Q}, <)$ has QE. By completeness $DLO \models Th(\mathbb{Q}, <)$, so the result follows.
(5) Show that DLO is not $\aleph_1$ categorical.
Solution: Define two models of DLO of size $\aleph_1$: the first is $\aleph_1$ but with copies of $\mathbb{Q}$ between any two α and $\alpha + 1 < \aleph_1$ and also a copy of $\mathbb{Q}$ below

0. The second is the same, but instead of a copy of $\mathbb{Q}$ below 0, put the same order again but in reverse ordering below 0. They are not isomorphic because the first one has no element with more than $\aleph_0$ elements below it.

Question 4.

Let K be an infinite field. Let $L = \{m_a \,| a \in k\} \cup \{0, +\}$ where m_a are unary functions, $+$ a binary function and 0 a constant. We let a K-vector space be a structure for L by interpreting $m_a(v) = a \cdot v$. Let T be the theory of an infinite K-vector space.

(1) Write down axioms for T.
(2) Show that T is λ-categorical for all $\lambda > |K| + \aleph_0$.
Solution: Let $V_1, V_2 \models T$ and $|V_1| = |V_2| = \lambda$. Let $B_1 \subseteq V_1$ and $B_2 \subseteq V_2$ be basis for V_1, V_2 resp. An easy calculation shows that $|V| = |B| + |K| + \aleph_0$ for any infinite vector space over K and a basis B. In our case it follows that $|B_1| = |B_2|$. So there is an isomorphism $f : V_1 \to V_2$, that extends any bijection between B_1 and B_2.
(3) Conclude that T is complete.
Solution: by Los-Vaught test.
(4) Show that if K is infinite then T is not $|K|$-categorical.
Solution: Let $V_1 = K$ (i.e. $\dim(V_1) = 1$). And $V_2 = K^2$.
(5) Show that if $V_1 \le V_2$ are two K-vector spaces, then $V_1 \prec V_2$.
Solution: Let λ be bigger than $|V_2| + \aleph_0 + |K|$. Let $V_2 \le V_3$ be of cardinality λ. By upwards Lowenheim-Skolem, there is some V_4 such that $V_2 \prec V_4$ and $|V_4| = \lambda$. Extending a basis of V_2 to a basis of V_4 and to a basis of V_3 gives us an isomorphism $f : V_3 \to V_4$ fixing V_2. This means that $V_2 \prec V_3$. By exactly the same argument, $V_1 \prec V_3$. Together, by question 1, we see that $V_1 \prec V_2$.