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MODEL THEORY – EXERCISE 12

To be submitted on Wednesday 6.07.2011 by 14:00 in the mailbox.

Definition. Let L be a signature.

(1) A theory T is said to be *absolutely categorical* if $M, N \models T \Rightarrow M \cong N$.
(2) A theory T is said to be λ-*categorical* for some cardinal λ if there exists a model of size λ and $M, N \models T, |M| = |N| = \lambda \Rightarrow M \cong N$.
(3) We write $M \prec N$ when M is an elementary substructure of N – if $\varphi(\bar{x})$ is a formula and $\bar{a} \in M$ then $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$.

Question 1.
Show that if $M_1 \subseteq M_2 \subseteq M_3$ are L-structures (M_1 is a substructure of M_2 and M_2 is a substructure of M_3) and $M_2 \prec M_3, M_1 \prec M_3$, then $M_1 \prec M_2$.

Question 2.
Let σ be a signature.

(1) Assume that σ is finite. Show that if M, N are two finite structures such that $M \equiv N$ then $M \cong N$.
Moreover, show that if M is a finite σ-structure, then there is a sentence φ such that $M \models \varphi$ and if $N \models \varphi$ then $N \cong N$.
(2) Now prove (1) (without the "moreover") for arbitrary L.
(3) Conclude that a theory T is absolutely categorical iff T is complete and has only finite models.

Question 3.
Let $L = \{<\}$ where $<$ is a binary relation symbol. Let DLO (in class it was denoted by $DLOWEP$) be the theory of densely ordered (between any two points there is another point) linear orders with no end points (i.e. there is no minimal or maximal element).

(1) Write down the axioms of DLO.
(2) Proof that DLO is $\aleph_0$-categorical.
Hints: assume that $M, N \models DLO$.
   (a) Suppose $f : A \to B$ is a map such that $|A| = |B|$ is finite, $A \subseteq M, B \subseteq N$ and f is an isomorphism (i.e. order preserving). Suppose $a \in M$, show that there is some $f' \supseteq f$ (i.e. extending f) such that $a \in Dom(f')$.
   (b) Suppose $f : A \to B$ is a map such that $|A| = |B|$ is finite, $A \subseteq M, B \subseteq N$ and f is an embedding (i.e. order preserving). Suppose $b \in N$, show that there is some $f' \supseteq f$ (i.e. extending f) such that $b \in Im(f')$.
   (c) Now, assume $|M| = |N| = \aleph_0$ and let $M = \{a_i \,|i < \omega\}, N = \{b_i \,|i < \omega\}$. Define a sequence of functions f_i such that
      - $Dom(f_i), Im(f_i)$ are finite.
      - $f_i : Dom(f_i) \to Im(f_i)$ is an isomorphism.
      - $a_i \in Dom(f_{2i+1}), b_i \in Im(f_{2i+2})$.
      - $f_i \subseteq f_{i+1}$.

   (d) Finish the proof.
(3) Deduce that DLO is complete.
(4) Prove that DLO has quantifier elimination (hint: use Exercise 4).
(5) Show that DLO is not $\aleph_1$ categorical.

Question 4.
Let K be an infinite field. Let $L = \{m_a \,|a \in k\} \cup \{0, +\}$ where m_a are unary functions, $+$ a binary function and 0 a constant. We let a K-vector space be a structure for L by interpreting $m_a(v) = a \cdot v$. Let T be the theory of an infinite K-vector space.

(1) Write down axioms for T.
(2) Show that T is λ-categorical for all $\lambda > |K| + \aleph_0$.
(3) Conclude that T is complete.
(4) Show that if K is infinite then T is not $|K|$-categorical.
(5) Show that if $V_1 \leq V_2$ are two K-vector spaces, then $V_1 \prec V_2$.