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# MODEL THEORY – EXERCISE 10

To be submitted on Wednesday 22.06.2011 by 14:00 in the mailbox.

**Definition.** We introduce the following notation. Let $L$ be a signature. Let $\Delta$ be a set of formulas.

(1) $M \Rrightarrow_{\Delta} N$ means: if $\varphi \in \Delta$ happens to be a sentence and $M \models \varphi$ then $N \models \varphi$. $M \equiv_{\Delta} N$ means that $M \Rrightarrow_{\Delta} N$ and $N \Rrightarrow_{\Delta} M$. If $\Delta$ contains all sentences then this just means $M \equiv N$.
(2) $f : M \to_{\Delta} N$ means that $f$ is a homomorphism from $M$ to $N$ and in addition if $M \models \varphi(\bar{a})$ for some tuple $\bar{a}$ from $M$, $N \models \varphi(f[\bar{a}])$.
(3) If $M$ is a structure and $A \subseteq M$ is a subset, we let $L(A)$ be $L$ with new constant symbols $c_a$ for elements $a \in A$. We denote $(M, A)$ the $L(A)$ structure we get interpreting the constants in the obvious way. $\Delta(A)$ comes from the formulas in $\Delta$ by replacing any tuple of free variables $\bar{x}$ by all possible tuples $\bar{c}_a$ of the same length from $A$.
(4) Given a structure $M$, the diagram of $M$, $D(M)$ is the $L(M)$ theory
$$\{\varphi(\bar{a}) \mid \varphi \text{ atomic or negaion}, \bar{a} \in M, M \models \varphi(\bar{a})\}.$$
(5) We say that a formula $\varphi$ is a $\exists\forall$ formula if it is of the form $\exists x_1 \dots \exists x_n \psi(x_1, \dots, x_n, \bar{y})$ where $\psi$ is universal.

**Question 1.**
Prove the following:

(1) $f : M \to_{\Delta} N$ iff $(M, M) \Rrightarrow_{\Delta(M)} (N, f[M])$ (here $(N, f[M])$ is an $L(M)$ structure where we interpret a constant $c_m$ as $f(m) \in N$).
(2) Let $\Delta$ be a set of sentences. Then $f : M \to_{\Delta} N$ iff $M \Rrightarrow_{\Delta} N$ and $f$ is a homomorphism from $M$ to $N$.
(3) $f : M \to N$ is a homomorphism iff $f : M \to_{at} N$ where $at$ is the set of all atomic formulas.
(4) If $\Delta$ contains $at$ and also the negation of all atomic formulas then $f$ is an embedding (see Ex. 1).
(5) A homomorphism $f : M \to N$ is injective iff $f : M \to_{\Delta} N$ for the set $\Delta = \{x \neq y\}$.
(6) If $\Delta$ is a set of sentences closed under negation, then $M \Rrightarrow_{\Delta} N$ implies $M \equiv_{\Delta} N$.
(7) If $\Delta$ is a set of formulas closed under negation, then $f : M \to_{\Delta} N$ implies that we have iff in definition (2) above.

**Question 2.**

(1) In class you proved that a theory $T$ is universal iff if $T$ is preserved under substructures. Show that the following statements are equivalent:
   (a) $T$ is existential, i.e. $T$ can be axiomatized by existential sentences.
   (b) $T$ is preserved under extensions, i.e. if $M \models T$ and $M \subseteq N$ then $N \models T$.

Hint for (b) implies (a): Show that for all $\varphi \in T$, there is a finite set of existential sentences $\varphi_i$ such that $T \models \bigvee \varphi_i$ and $\models \bigvee \varphi_i \to \varphi$. Use compactness.
Solution: Suppose we did the hint, then for all $\varphi$, let $\psi_\varphi$ be this disjunction. Then $\psi_\varphi$ is existential. Let $\Sigma = \{\psi_\varphi \,|\varphi \in T\}$. Then $\Sigma \equiv T$.
Let $\Gamma = \{\alpha \,|\alpha$ is existential and $\alpha \models \varphi\}$. If there is a finite subset $\Gamma_0$ of $\Gamma$ such that $T \models \bigvee \Gamma_0$, then we are done. If not, by compactness, we find a model $M$ such that $M \models T$, and $M \models \neg\alpha$ for all $\alpha \in \Gamma$. By (1), $D(M) \models \varphi$ ($N \models D(M)$ iff $M$ is embeddable in $N$ iff $M$ is isomorphic to a substructure of $N$). So there is a sentence $\psi(\bar{a}) \in D(M)$ (so it is quantifier free) such that $\psi(\bar{a}) \models \varphi$. But since $\bar{a}$ are new constants, this means that $\exists \bar{x}\psi(\bar{x}) \models \varphi$ (look at Ex. 3, 1, (4)). But then $\exists \bar{x}\psi(\bar{x}) \in \Gamma$, and this sentence holds true in $M$ – a contradiction.

(2) Conclude that the theory of Groups is neither existential nor universal.
Solution: it is not universal, because for instance $\mathbb{N}$ is a substructure of $\mathbb{Z}$. The fact that it is not existential is much easier.

**Question 3.**

(1) Suppose $\varphi$ is a $\exists\forall$ sentence in the signature $L = \{<\}$, where $<$ is a binary relation symbol. Show that if $\varphi$ is true in $(\mathbb{R}, <)$ then it is also true in $(\mathbb{N}, <)$.
Solution: Suppose $\varphi = \exists \bar{x}\forall \bar{y}\alpha(\bar{x}, \bar{y})$. There are $\bar{r} \in \mathbb{R}$ witnessing $\mathbb{R} \models \varphi$. Use an automorphism to move them to natural numbers.

(2) Is the converse true as well?
Solution: no. $\mathbb{N}$ thinks there's a minimal element.

**Question 4.**
Let $L$ be some signature.
In Exercise 2, Question 3, (3), you proved that if $\varphi$ is equational then $\varphi$ is preserved under homomorphic images, products, and substructures (and you used this question in Exercise 9, Question 3). Now prove that if $\varphi$ is preserved under homomorphic images, products, and substructures then $\varphi$ is equivalent to a conjunction of equational sentences.

(1) Let $\Sigma$ be the set of equational sentences $\psi$ such that $\varphi \models \psi$. Show that is enough to show that $\Sigma \models \varphi$.
Solution: By compactness, there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$, so $\varphi \equiv \bigwedge \Sigma_0$.

(2) Let $A$ be any model of $\Sigma$. Show that it is a homomorphic image of a substructure of a product of models of $\varphi$. Use the following steps:
  (a) Show that there is a set $C$ of structures, such that if $M \models \varphi$ and $M$ is generated by finitely many elements, then there is some $M' \in C$ such that $M' \cong M$.
  Solution: The size of a structure generated by finitely many elements is at most $\kappa = |L| + \aleph_0$. Let $C$ be the set of all structures such that their universe is a subset of $\kappa$, and such that they are a model of $\varphi$.
  (b) Let $D$ be the set of pairs $(M, f)$ where $M \in C$ and $f : A \to M$ some function. Let $\mathfrak{M} = \prod \{M \,|(M, f) \in D\}$ (so there is some repetition). Define $F : A \to \mathfrak{M}$ by $F(a)(M, f) = f(a)$. Let $\mathfrak{N}$ be the structure

generated by the image of $F$. Deduce that $\mathfrak{N} \models \varphi$.
Solution: $\mathfrak{N}$ is a substructure of a $\mathfrak{M}$ which is a model of $\varphi$ by assumption.

(c) Define a homomorphism $G : \mathfrak{N} \to A$ that satisfies $G(F(a)) = a$ and conclude.
Solution: If we see this then $A$ is a homomorphic image of $\mathfrak{N}$ so we are done. Assume that $R \in L$ is some $n$-ary relation symbol, $t_i(x_1, \ldots, x_m)$ are terms, $a_1, \ldots, a_m \in A$ and
$\mathfrak{N} \models R(t_1(F(a_1), \ldots, F(a_m)), \ldots, t_n(F(a_1), \ldots, F(a_m)))$. So for all
$(M, f) \in D$, $M \models R(t_1(f(a_1), \ldots, f(a_m)), \ldots, t_n(f(a_1), \ldots, f(a_m)))$.
Now, if $\varphi \not\models \forall \bar{x} R(t_1(\bar{x}), \ldots, t_n(\bar{x}))$, then there is some model $M \models \varphi$ such that $M \models \exists \bar{x} \neg R(t_1(\bar{x}), \ldots, t_n(\bar{x}))$. Suppose $c_1, \ldots, c_m \in M$ witness this, and let $M_0$ be the substructure of $M$ generated by $c_1, \ldots, c_m$. Then we may assume that $M_0 \in C$. Let $f : A \to M_0$ be $f(a_i) = c_i$. Then $(M, f) \in D$ and we get a contradiction. It follows that $\varphi \models \forall \bar{x} R(t_1(\bar{x}), \ldots, t_n(\bar{x}))$ so this sentence is in $\Sigma$, so $A$ satisfies it, and hence $G$ is a homomorphism. (Note that this proof also proves that $G$ is well defined).