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MODEL THEORY – EXERCISE 8

To be submitted on Wednesday 08.06.2011 by 14:00 in the mailbox.

Definition.

(1) For a set X and a number $n < \omega$, let $X^{[n]}$ be the set of subsets of X of size n.
(2) A graph is a structure $G = (V, E)$ where V is a set of vertices and the edge relation E is a binary relation which is symmetric ($xEy \to yEx$) and anti-reflexive ($\neg xEx$).
(3) The Finiteness Theorem / The Compactness Theorem: if Σ is a set of sentence such that each finite subset of it is consistent (has a model) then Σ has a model.
(4) Let $n < \omega$. We say that a graph G is n-colorable, if there exists a function $C : V \to n$ such that if $a, b \in V$ and aEb then $C(a) \neq C(b)$.
(5) A class of L-structures K is called elementary if there exists a set of sentences Σ such that $K = Mod(\Sigma)$.

Question 1.

(1) Let $L = \{<\}$. Show that the class of well orderings is not elementary.
Solution: Recall that $<$ is a well order iff there is no infinite descending chain. If there were such a Σ, then consider $\Sigma \cup \{c_{i+1} < c_i\}$ where c_i are new constants. Then this set is finitely satisfiable (because $(\mathbb{N}, <)$ is always a model with some choice of elements for c_i), but not satisfiable – contradiction.
(2) Show that the class of all finite sets (in the signature $L = \approx$) is not elementary.
(3) Let $L = \{+, \cdot, 0, 1, <\}$, and let $T = Th(\mathbb{N}, +, \cdot, 0, 1, <)$. Show that there exists a model M of T with an element c which is greater than all natural numbers (i.e. $c > 1^M, (1+1)^M$ etc.)
(4) Show that in the model constructed in (3), there is no minimal such c.
Solution: For any c, $c-1$ is well defined (it's the only x satisfying $x+1 = c$), and $c-1$ is also bigger than $\mathbb{N}$, because otherwise, if $c-1 < k$, then $c < k+1$.
(5) Let $T = Th(\mathbb{R}, +, \cdot, 0, 1, <)$. Show that there is a model $M \models T$ with an element $0 < \varepsilon \in M$ which is infinitesimal: for every positive integer n, $\varepsilon < \left(1/(1 + \ldots + 1)^M\right)$ where the 1 is summed n-times.

Question 2.

The infinite Ramsey Theorem states as follows: Suppose V is an infinite set and $C : V^{[2]} \to \{0, 1\}$. Then there exists an infinite subset $U \subseteq V$ and $i \in \{0, 1\}$ such that $C(\{x, y\}) = i$ for all $x, y \in U$ (in other words, $C \restriction U$ is constant).
You may think of C as a coloring function (of pairs from V), and then U is monochromatic.
The finite Ramsey Theorem states as follows: For all $k < \omega$ there exists some $n < \omega$

such that if $|V| = n$, and $C : V^{[2]} \to \{0,1\}$ then there exists some $U \subseteq V$ of size k which is monochromatic.
Remark: this is actually the Ramsey Theorem for coloring of pairs in 2 colors.

(1) Prove the infinite Ramsey theorem.
Solution: Construct a sequence of elements $a_i \in V$, sets $V_i \subseteq V$ and $\varepsilon_i \in \{0,1\}$ for $i < \omega$ such that $V_{i+1} \subseteq V_i$, $a_i \in V_i \backslash V_{i+1}$,$V_i$ is infinite, $C(\{a_i, u\}) = \varepsilon_i$ for all elements u from V_{i+1}. If we succeed, then there exists some $i_0 \in \{0,1\}$ such that $\varepsilon_i = i_0$ for infinitely many $i < \omega$, and then let $U = \{a_i \,|\varepsilon_i = i_0\}$.
Construction: let $V_0 = V$, $a_0 = b_0$. For some i_0, and infinitely many $b \in V$, $C(\{a_0, b\}) = i_0$. Let $\varepsilon_0 = i_0$. Suppose we chose a_i, V_i and ε_i for $i \leq n$ such that everything above holds, and in addition $C(\{a_n, b\}) = i'$ for some i' and infinitely many $b \in V_n$. Then let V_{n+1} be this infinite set, and let a_{n+1} be some element from it. For infinitely many $b \in V_{n+1}$, $C(\{a_{n+1}, b\})$ is constant, so we can continue.

(2) Deduce the finite Ramsey Theorem from the infinite one using the Compactness Theorem.
Solution: Let k be given. Let A be a set of constants, and let $L_A = A \cup \{C\}$ where C is a binary relation. Let T_A be the theory axiomatized by $\{a \neq b \,|a \neq b \in A\}$, C is symmetric and anti-reflexive (so an infinite graph), and for every $\{a_0, \ldots, a_{k-1}\} \subseteq A$, the sentence

$$\bigvee_{s_1 < s_2 < k, t_1 < t_2 < k} \left(C\left(a_{s_1}, c_{s_2}\right) \wedge \neg C\left(c_{t_1}, c_{t_2}\right)\right).$$

I claim that T_A is consistent iff there is a coloring $C : A^{[2]} \to \{0,1\}$ with no monochromatic set of size k: Given a model M of T_A, define $C(\{a,b\}) = 1$ iff $C^M(a,b)$. On the other hand, if there is such a coloring C, define $M = A$ with $C^M(a,b)$ iff $C(\{a,b\}) = 1$.
Let A be $\{c_i \,|i < \omega\}$. By the infinite Ramsey, T_A is not consistent, so there is some n such that if $A_0 = \{c_i \,|i < n\}$ then T_{A_0} is inconsistent, i.e. every coloring of $A_0^{[2]}$ has a monochromatic subset of size k (and so all sets of size n).
Remark: there exists a proof that uses only induction on natural numbers.

Question 3.
Show that if $G = (V, E)$ is an infinite graph such that every finite sub graph of it is n-colorable then G is n-colorable.
Solution: Let $L = \{c_a \,|a \in V\} \cup \{f\} \cup \{d_0, \ldots, d_{n-1}\}$ where c, d are constants and f is a unary function symbol. Let T be the theory saying that $c_a \neq c_b$ for $a \neq b$, $f(x) \in \{d_0, \ldots, d_{n-1}\}$ for all x. For all a, b such that aEb, add a sentence $f(c_a) \neq f(c_b)$. Then T has a model iff G is n-colorable. If T is inconsistent, then some finite subset of it is already inconsistent, i.e. there is some finite $V_0 \subseteq V$ such that if $G_0 = (V_0, E \restriction V_0)$ then T_{G_0} is not consistent. But then G_0 is not n-colorable – contradiction.

Question 4.
Show that the following are equivalent:

(1) The Compactness Theorem.

(2) Let T_1, T_2 be sets of L-sentences. Assume that for every L-structure M, M is a model of T_1 iff M is not a model of T_2. Then there are some finite $\Sigma_1 \subseteq T_1$, $\Sigma_2 \subseteq T_2$ such that $\Sigma_1 \equiv T_1$, $\Sigma_2 \equiv T_2$ (i.e. $\Sigma_1 \models T_1$, $\Sigma_2 \models T_2$).
Solution: (1) to (2) The assumption says that $T_1 \cup T_2$ is inconsistent. So there is some $\Sigma_1 \subseteq T_1, \Sigma_2 \subseteq T_2$ such that $\Sigma_1 \cup \Sigma_2$ is inconsistent. If $M \models \Sigma_1$ then $M \not\models \Sigma_2$, so it cannot be that $M \models T_2$, so $M \models T_1$, i.e. $\Sigma_1 \models T_1$. For Σ_2, T_2 it's the same.

(3) Let T_1, T_2 be sets of L-sentences. Assume that T_2 is finite and that $T_1 \equiv T_2$. Then T_1 is finitely axiomatizable (i.e. there is some finite $\Sigma \subseteq T_1$ such that $\Sigma \equiv T_1$).
Solution: (2) to (3): let $\alpha = \bigwedge T_2$. Then for every structure M, M is a model of T_1 iff $M \models \alpha$ iff M is not a model of $\neg\alpha$. By (1), there is some finite Σ_1 equivalent to T_1.
(3) to (1): Assume that Σ is a set of sentences with no model. Then $\Sigma \equiv \{\bot\}$ (where $\bot$ is always interpreted as false, can be replaced by $\forall x\,(x \neq x)$). By (2), there is some finite $\Sigma_0 \subseteq \Sigma$ which is inconsistent.