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MODEL THEORY – EXERCISE 7

To be submitted on Wednesday 01.06.2011 by 14:00 in the mailbox.

Definition.

(1) Let A be a set, and $F \subseteq \mathcal{P}(A)$ is called a *filter* on A if the following holds:
  (a) $\emptyset \notin F$.
  (b) $F \neq \emptyset$.
  (c) If $X, Y \in F$ then $X \cap Y \in F$.
  (d) If $X \in F$ and $X \subseteq Y \subseteq A$ then $Y \in F$.

(2) F is said to be a *principle filter* if there exists some $X \subseteq A$ such that $F = \{Y \subseteq A \mid Y \supseteq X\}$.

(3) F is said to be an *ultrafilter* if for every $X \subseteq A$, $X \in F$ or $A \backslash X \in F$.

(4) Let A be an infinite set. A family $C \subseteq \mathcal{P}(A)$ is called *independent* if for every $n, m \in \mathbb{N}$ and $X_1, \ldots, X_n \in C$, $Y_1, \ldots, Y_m \in C$ such that no two are equal, $\bigcap_i X_i \cap \bigcap (A \backslash Y_j) \neq \emptyset$.

(5) Recall that given a structure M for a signature L and a subset $A \subseteq M$, $L(A)$ is the signature L where we add constants $\{c_a \mid a \in A\}$, and we interpret them as follows: $c_a^M = a$.

(6) A structure M is said to be *countably saturated* if given a set $\{\varphi_n(x) \in L(M) \mid n < \omega\}$ such that each finite subset has a solution a_n, then $\{\varphi_n(x) \mid n < \omega\}$ has a common solution a.

Question 1.

(1) Suppose F is a filter on a set A and there exists some finite $X \subseteq A$ such that $X \in F$. Show that F is principle.

(2) Show that if F is a principle ultrafilter on a set A then there exists $a \in A$ such that $F = \{X \in A \mid X \supseteq \{a\}\}$.

(3) Suppose F is a principle ultrafilter on a set A, and that M_i for $i \in A$ is an L-structure. Show that the ultra-product $\prod_{i \in A} M_i / F$ is isomorphic to $M_{i_0}$for some $i_0 \in A$.

(4) Suppose A is an infinite set. Let I be the set of all finite subsets of A. For each $s \in I$, let $X_s = \{t \in I \mid t \supseteq s\}$. Show that $\{X \subseteq A \mid \exists s \in I\, (X \supseteq X_s)\}$ is a filter on A.

Question 2.

Let A be an infinite set. Compute the cardinality of the set $U = \{D \mid D \text{ is an ultrafilter on } A\}$. Use the following steps:

(1) Show that $|U| \leq 2^{2^{|A|}}$.

(2) Show that there exists a family $C \subseteq \mathcal{P}(A)$, such that $|C| = 2^{|A|}$ and if $X_1 \neq X_2 \in C$ then $X_1 \subsetneq X_2$, $X_2 \subsetneq X_1$.
Hint: look at graphs of functions, use the fact that $|A|^2 = |A|$.

(3) Show that there exists an independent family $C \subseteq \mathcal{P}(A)$ of subsets of A of cardinality $2^{|A|}$.
Hint: let C_0 be as above. Let $M = \omega \times A^{<\omega}$ (where $A^{<\omega}$ is the set of finite

sequences from A). For all $X \in C_0$, let $\bar{X}$ be the set of pairs $(n, \bar{a})$ from M such that: the length of $\bar{a}$ is divisible by n, so $\bar{a} = \bar{a}_0 \frown \ldots \frown \bar{a}_{k-1}$ where $\lg(\bar{a}_i) = n$, and there is $i < k$ such that all elements from $\bar{a}_i$ are in X. Show that $\left\{ \bar{X} \mid X \in C_0 \right\}$ is an independent set in M.

(4) Conclude that $|U| = 2^{2^{|A|}}$.

Question 3.

(1) Suppose A is an infinite set, and D is a filter on an infinite set I. Show that $|A| \leq \left| A^I / D \right| \leq |A|^{|I|}$.

(2) Show that for every A and I there exists an ultrafilter on I such that $\left| A^I / D \right| = |A|$.

(3) Let D be a non-principle filter on ω. Show that $|\mathbb{N}^\omega / D| = 2^{\aleph_0}$.
Hint: use the fact that every natural number has a unique presentation in base 2.

Question 4.

Prove the following:

Let $\{A_i \mid i \in \mathbb{N}\}$ be structures in a signature L. Let D be an ultrafilter containing the filter of co-finite sets, and let A be the ultraproduct. Then A is countably saturated.

Follow these steps:

(1) We are given a set $\{\varphi_n(x) \in L(A) \mid n < \omega\}$ as in the definition, and we want to find a common solution. Show that it's enough to show it assuming $\varphi_n \in L$ (i.e. no new constant symbols from A that do not appear in L).

(2) For $n < \omega$, pick $a_n \in A_n$ such that $A_n \models \varphi_1(a_n), \ldots, \varphi_k(a_n)$ for $k \leq k(n)$, where $k(n) \leq n$ is as large as possible (can be 0). Show that $\langle a_n \mid n \in \omega \rangle / D$ is a common solution.