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MODEL THEORY – EXERCISE 6

To be submitted on Wednesday 25.05.2011 by 14:00 in the mailbox.

Definition.

(1) For a set of ordinals, $s \subseteq \mathbf{On}$, let the order type of s, $\mathrm{otp}(s)$ be the unique ordinal with which s is isomorphic.
(2) Suppose α is a ordinal. A subset $B \subseteq \alpha$ is called *cofinal in* α (or unbounded) if $\alpha = \bigcup B$ (this means that for every $\beta < \alpha$, there is some $\gamma \in B$ such that $\beta < \gamma$).
(3) For an ordinal α, the *cofinality* of α, $\mathrm{cf}(\alpha)$, is $\min\{\mathrm{otp}(B) \,|B \subseteq \alpha \text{ is cofinal}\}$.
(4) A cardinal λ is called *regular* if $\mathrm{cf}(\lambda) = \lambda$.

Question 1.

Prove the Cantor-Bernstein theorem: if A, B are sets, and there is some injective function $f : A \to B$ and some injective function $g : B \to A$, then $|A| = |B|$.
Solution: use the well-order principle. By this we may assume that both A and B are cardinal numbers, κ, λ. First show that if $s \subseteq \alpha \in \mathbf{On}$, then the unique ordinal β which is order-isomorphic to s is $\leq \alpha$: Suppose $h : s \to \beta$ is an isomorphism, show by induction on $\gamma \in s$ that $h(\gamma) \leq \gamma$. Then it follows that $h[s] = \beta = \bigcup_{\gamma \in s} h(\gamma) \leq \alpha$. Let $s = g[\lambda] \subseteq \kappa$, then s is order isomorphic to some $\gamma \leq \kappa$, so $\lambda = |s| \leq \gamma \leq \kappa$. Similarly, $\kappa \leq \lambda$.
Remark: there exists also a proof that does not use the axiom of choice at all, it is a little bit more complicated.

Question 2.

Let α be an ordinal.

(1) Show that if α is a successor ordinal, then $\mathrm{cf}(\alpha) = 1$.
Solution: the cofinal set is the last element.
(2) Show that $\mathrm{cf}(\alpha)$ is always a cardinal.
Solution: By (1) we may assume that α is limit. Let B be cofinal in α with $\beta := \mathrm{otp}(B) = \mathrm{cf}(\alpha)$. Suppose $\gamma < \beta$ is a <u>cardinal</u>, and there is some isomorphism $f : \gamma \to B$. Define (recursively) $h : \gamma \to B \cup \{\alpha\}$ by
$h(i) = \min\{b \in B \,|\forall j < i\,(b > f(j), h(j))\} \cup \{\alpha\}$. Obviously, h is well defined. If $i < j < \gamma$ and $h(i) \neq \alpha$ then $h(i) < h(j)$. By definition, $\{i < \gamma \,|h(i) \neq \alpha\}$ is an initial segment of α, so it is an ordinal, $\gamma' \leq \gamma$, and $h \restriction \gamma'$ is an order isomorphism. If $\gamma' < \gamma$, then B is bounded by $B' = h[\gamma'] \cup f[\gamma']$, but $|B'| = |\gamma'| + |\gamma'| < \gamma$, so $\mathrm{otp}(B') < \gamma$ – contradiction to the choice of β. Hence $\gamma' = \gamma$, and so h is an order isomorphism from γ onto $h[\gamma]$, and obviously, $h[\gamma]$ bounds B so also α – contradiction to the choice of β.
(3) Conclude that $\mathrm{cf}(\alpha)$ can be defined by $\min\{|B| \,|B \subseteq \alpha \text{ is cofinal}\}$.
Solution: Obviously, $\mathrm{cf}(\alpha)$ is at least this cardinal. On the other hand, if $|B| < \mathrm{cf}(\alpha)$, then $\mathrm{otp}(B) < \mathrm{cf}(\alpha)$, so B does not bound α.

Now let λ be an infinite cardinal.

(4) Show that $\lambda \geq cf(\lambda)$.
Solution: obviously, λ is unbounded in λ.
(5) Show that $cf(cf(\lambda)) = cf(\lambda)$.
Solution: By (1) $cf(cf(\lambda)) \leq cf(\lambda)$. Suppose $B \subseteq cf(\lambda)$ is of size $< cf(\lambda)$ and unbounded. There is some $C \subseteq \lambda$ of order type $\mathrm{cf}(\lambda)$ and unbounded. Let $f : \mathrm{cf}(\lambda) \to C$ be an order isomorphism. Then $f[B] \subseteq C$ is unbounded in λ. But then $|f[B]| < \mathrm{cf}(\lambda)$ – a contradiction.
(6) Show that $\aleph_0$ is regular, and that κ^+ is regular for all infinite κ.
Solution: for $\aleph_0$ - it is clear that there is no finite bounding subset. For κ^+: if $B \subseteq \kappa^+$ is bounding, and $|B| \leq \kappa$, then, as $\kappa^+ = \bigcup B$, and for all $\alpha \in B$, $|\alpha| \leq \kappa$, $|\bigcup B| \leq \sum \{|\alpha| \,| \alpha \in B\} \leq \kappa \cdot \kappa = \kappa$.
(7) Show that for limit ordinal α, $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$ and find an irregular cardinal.
Solution: $\{\aleph_i \,| i < \alpha\}$ is cofinal in $\aleph_0$. So $\aleph_\omega$ is irregular.

Question 3.

(1) Let λ be a cardinal. Show that if $\langle \kappa_i \,| i < \lambda\rangle$ is a sequence of λ cardinals, such that $i < j < \lambda \Rightarrow \kappa_i < \kappa_j$, then $\sum_{i<\lambda} \kappa_i < \prod_{i<\lambda} \kappa_i$ (where $\sum_{i<\kappa} \kappa_i$ is the cardinality of the disjoint union $\coprod \kappa_i$, and $\prod \kappa_i$ is the cardinality of the Cartesian product).
Hint: try to find a diagonalizing argument, as in the proof of $\kappa < 2^\kappa$. Note that for $i < \lambda$, $\sum_{j \leq i} \kappa_i = \kappa_i$.
Solution: Let $\lambda_1 = \sum_{i<\lambda} \kappa_i$, $\lambda_2 = \prod_{i<\lambda} \kappa_i$. Obviously, we have $\lambda_1 \leq \lambda_2$ (the injective function which takes $\alpha < \kappa_i$ to $g : \lambda \to \bigcup \kappa_i$ where $g(j) = 0$ for all $j \neq i$ and $g(i) = \alpha$ shows it). On the other hand, if there was also a function $h : \coprod \kappa_i \to \prod \kappa_i$ which is surjective, we shall get a contradiction. For $i < \lambda$,

$$\kappa_i \leq \sum_{j \leq i} \kappa_i \leq |i| \cdot \kappa_i \leq \kappa_i \cdot \kappa_i = \kappa_i$$

($|i| < \kappa_i$ because κ_i is increasing). For $i < \lambda$, let $\alpha_{i+1} < \kappa_{i+1}$ be the first element in κ_{i+1} such that α_{i+1} does not appear in $\pi_{i+1}\left(h\left(\coprod_{j \leq i} \kappa_i\right)\right)$ (where π_{i+1} is the projection). It exists because the cardinality of $\coprod_{j \leq i} \kappa_j$ is $\kappa_i < \kappa_{i+1}$. For $i < \lambda$ limit, let $\alpha_i = 0$. Consider $g = \langle \alpha_i \,| i < \lambda\rangle$. Then $g = h(\alpha)$ for some $\alpha < \kappa_i$ for some i. But then α_{i+1} is different than $\pi_{i+1}(h(\alpha))$ – contradiction.
(2) Conclude that $\mathrm{cf}(2^\kappa) > \kappa$ (note that this generalizes the fact proved in class that $2^\kappa > \kappa$).
Hint: deal with 2 cases: 2^κ is a successor or limit cardinal.
Solution: If 2^κ is a successor, then $\mathrm{cf}(2^\kappa) = 2^\kappa$ by Question 1, (6), and in that case we know this from class. So we assume that 2^κ is limit. In that case, if $\bigcup B = 2^\kappa$ (i.e. B is unbounded), then $\{|\alpha| \,| \alpha \in B\}$ is also unbounded. Hence, if $|B| \leq \kappa$, then we have a sequence of increasing cardinals of length κ. But then

$$\sum B < \prod B \leq (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$$

– a contradiction.

Question 4.
Let K be a field.

(1) Show that the cardinality of the algebraic elements over K in some field extension F is bounded by $|K| + \aleph_0$.
Solution: Very similar to (2).
(2) Let V be an infinite vector space over K, and let B be a basis for V. Show that $|B| + |K| + \aleph_0 = |V|$.
Solution: We easily have $\leq$. On the other hand, there is a surjective function from $C \times D$ to V, where C is the set of all finite sequences from B, and D is the set of all finite sequences from K. The function is: given $\langle b_1, \ldots, b_n \rangle$ and $\langle c_1, \ldots, c_k \rangle$, take it to $\sum_{i=1}^{\min\{n,k\}} c_i b_i$. By a theorem taught in class, $|C| = |B| + \aleph_0$and $|D| = |K| + \aleph_0$.
(3) Show that the cardinality of the irrational real numbers is $2^{\aleph_0}$.
(4) Show that the number of the real transcendental elements is $2^{\aleph_0}$ (i.e. elements that are in $\mathbb{R}$ but not algebraic over $\mathbb{Q}$).