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MODEL THEORY – EXERCISE 5

To be submitted on Wednesday 18.05.2011 by 14:00 in the mailbox.

Definition.

Axioms of set theory (ZFC with classes):
We work in two sorted signatures, i.e. we have 2 kinds of variables, and 2 kinds of objects (so a structure contains two disjoint parts). One type of objects being *sets* and the other type of objects being *classes*, with the element-relation $\in$ defined between sets and sets and between sets and classes only (so not between classes and sets). Use lower case letters as variables for sets and capital letters for classes. Here are the axioms

(1) Axioms about sets:
 (a) Extensionality: Sets containing the same elements are equal.
 (b) Empty set: The empty set exists.
 (c) Pairing: For any sets a and b, $\{a, b\}$ is a set. This means that there is a set which has exactly the elements a and b.
 (d) Union: For every set a, the union $\bigcup a = \{z \mid \exists y \in a\,(z \in y)\}$ is a set.
 (e) Power Set: For every set a, the power set $\mathcal{P}(a) = \{y : y \subseteq a\}$ is a set.
 (f) Infinity: There is an infinite set.
 (g) Regularity: Every nonempty set has an $\in$-minimal element.
 (h) Choice: if a if a set such that $\emptyset \notin a$ then there exists a function (which is a set) $f : a \to \bigcup a$ such that $f(x) \in x$ for all $x \in a$. See 2(c) below for explanation of functions.

(2) Axioms about classes
 (a) Class extensionality: Classes containing the same elements are equal.
 (b) Comprehension: If $\varphi(x; y_1, \ldots, y_m; Y_1, \ldots, Y_n)$ is a formula in which only set-variables are quantified, and if $b_1, \ldots, b_m; B_1, \ldots, B_n$ are sets and classes, respectively, then $\{x \mid \varphi(x; b_1, \ldots, b_m; B_1, \ldots, B_n)\}$ is a class.
 (c) Replacement: If a class F is a function, i.e. if for every set b there is a unique set $c = F(b)$: such that $(b, c) = \{\{b\}, \{b, c\}\}$ belongs to F, then for every set a the image $\{F(z) \mid z \in a\}$.
 Note: this implies that if a is a set, then, using the notation of (b), $\{x \in a \mid \varphi(x; b_1, \ldots, b_m; B_1, \ldots, B_n)\}$ is a set (use the function $F(b) = b$ if b satisfies the formula, and if not, then $F(b)$ is some element in a satisfying the formula (if there is none, then this set is empty), and then this set is the image of a).

More definitions:

(1) A set (or class) a is called *transitive* if $\forall x \in a \forall z \in x\,(z \in a)$.
(2) A *well ordering* is a set (or class) a with a binary relation on it $<$ (so $< \subseteq a \times a$) which is also a set (or a class if a is) such that $(a, <)$ is a linear order and for all non-empty $x \subseteq a$ (x is a set), there exists $x_0 \in x$ which is $<$ minimal.

(3) A set α is called an *ordinal* if it is transitive and $\in$ is a well order on α. The class of all ordinals is denoted by **On**.
(4) The *recursion theorem* is: Let $(X, <)$ be a well ordered class or set, such that for each $a \in X$, $X_a = \{x \in X \,| x < a\}$ is a set. Let W be a class (or set). For $a \in X$, let $F_a = \{f : X_a \to W\}$ be the class (or set if X, W are sets) of functions from X_a to W. Let $F : \bigcup_{a \in X} F_a \to W$. Then there is a unique function $f : X \to W$ that satisfies $f(a) = F(f \restriction X_a)$. If X is a set, then f is a set, and if X is a class, then f is a class.

Question 1.
This question is only for students who do not feel confident about writing formulas yet. It is not obligatory.
Let $L = \{\in\}$ be the signature of set theory, i.e. $\in$ is a binary relation symbol and we write $x \in y$ instead of $\in(x, y)$. Of course, $\approx$ is also in L. We also have classes as above, which will be denoted using capital letters. Write down the axioms of set theory as stated above in L. Note that axioms 2(b) and 2(c) are in fact a scheme of axioms. You may use notations from Exercise 4.

All questions below are about set theory.

Question 2.

(1) Show that the following is equivalent for a set or class a:
  (a) a is transitive.
  (b) $x \in a \Rightarrow x \subseteq a$.
  (c) $x \subseteq a \Rightarrow \bigcup x \subseteq a$.
(2) Show that if α is an ordinal then $\alpha \cup \{\alpha\}$ is also an ordinal.
(3) Show that if $\alpha \in \mathbf{On}$ and $x \in \alpha$ then $x \in \mathbf{On}$.
(4) Show that if $\alpha \in \mathbf{On}$ and $\beta \subseteq \alpha, \beta \neq \alpha$, is an ordinal, then $\beta \in \alpha$.
Solution: let $\alpha' \in \alpha$ be the first ordinal in α that is not in β. If $\beta' \in \beta$ then $\beta' \in \alpha$, and if $\beta' \notin \alpha'$ then $\alpha' \in \beta'$, so $\alpha' \in \beta$. So $\beta \subseteq \alpha'$. On the other hand, if $\alpha'' \in \alpha'$ then by definition, $\alpha'' \in \beta$, so we have $\alpha' = \beta$.
(5) Prove that a set or class a with an order $< \subseteq a^2$ is well ordered iff there is no sequence $\langle x_i \in a \,| i \in \mathbb{N}\rangle$ such that $x_{i+1} < x_i$ for all i. (Note: a sequence is a function, so in this case, it's a function from $\mathbb{N}$ to a).
Solution: if there is such a sequence, then the set $\{x_i \,| i \in \mathbb{N}\}$ exists by replacement, and it doesn't have a minimal element. On the other hand, if a is not well ordered, let $a_0 \subseteq a$ be a set without a minimal element. Then for each $x \in a_0$ there is $y \in a_0$ such that $x < y$. The axiom of Choice implies that there is a function $f : a_0 \to a_0$ such that $f(x) < x$ for all x. Let $g : \mathbb{N} \to a_0$ be $g(n) = f(\ldots f(x))$ where the composition is n times, so $a_i = g(i)$. This last argument uses the recursion theorem from class.
(6) Show that **On** is a well ordered class (with $\in$ as the ordering). Remember that you must also show that $\in$ is a linear ordering.
Solution: First of all $\in$ is a linear ordering on ordinals: if $\alpha \notin \beta$ and $\beta \notin \alpha$, look at $\gamma = \alpha \cap \beta$. This is also an ordinal, and $\gamma \neq \alpha, \beta$ (by 4). So $\gamma \in \alpha$ and $\gamma \in \beta$ (again by 4), but then $\gamma \in \gamma$ – contradiction. It is a well ordering: if not, there is a sequence as in (5), and this sequence is contained in one ordinal, which is well ordered.

(7) Conclude that there is no set x such that x contains exactly all ordinals (i.e. show that **On** is not a set).
Solution: x is itself an ordinal by (6). But then $x \in x$ – a contradiction.

Question 3.
Show that the following is equivalent (modulo all the axioms of set theory except the axiom of choice):

(1) The axiom of choice.
(2) For every non-empty set a, there exists some binary relation $< \subseteq a^2$ such that $(a, <)$ is a well order.
(3) Zorn's lemma: If $(a, <)$ is a non-empty partially ordered set, such that if $a_0 \subseteq a$ is a chain (i.e. it is linearly ordered by $<$), then there is some element greater than a_0, then there is some maximal element $b \in a$, i.e. such that $\forall x \in a\, (\neg (b < x))$.
Hints: for (1) implies (2): use the recursion theorem to get an injective function from an ordinal onto a (note exactly where you use choice). For (2) implies (3): assume that (3) is false, and again use the recursion theorem to find a function $f : \mathbf{On} \to a$ which is increasing, i.e. $\alpha < \beta \Rightarrow f(\alpha) < f(\beta)$, if this could be done, then it would follow that **On** is a set. For (3) implies (1): consider the set of partial functions.
Solution: (1) implies (2): Since a is not empty, there is some $x \in a$.
Use the recursion theorem: Let $X = \mathbf{On}$, $W = a$. Let

$$b = \{W \backslash \operatorname{image}(g) \,|\, a \in X, g : X_a \to W \text{ not onto}\}.$$

By choice, there is some $h : b \to \bigcup b$ such that $h(a \backslash \operatorname{image}(g)) \in a \backslash \operatorname{image}(g)$. Now, for $g \in F_\alpha$, let $F(g)$ be x if g is onto and if not, $F(g) = h(a \backslash \operatorname{image}(g))$. By the recursion theorem, there is some class function $f : \mathbf{On} \to a$ such that $f(\alpha) = F(f \restriction \alpha)$. If for all α, $f \restriction \alpha$ is not onto, then f is injective (by choice of F). But that is impossible, because then **On** is a set: it is the image of the inverse of f. So let α_0 be the first ordinal such that $f \restriction \alpha_0$ is onto. Then $f \restriction \alpha_0$ is injective, so f induces a well order on a.
(2) implies (3): Assume there is no maximal element. Suppose $\prec$ is a well ordering on a, and choose some $x \in a$. Let $f : \mathbf{On} \to a$ be defined by recursion ($X = \mathbf{On}$, $W = a$): if $g \in F_\alpha$ is such that g is not $<$ increasing, then let $F(g) = x$, if g is increasing, then image(g) is a chain, so let $F(g)$ be the $\prec$-first element that is $<$-greater than the image of g (there is one such element, because there is some element that is $\leq$- greater than the image of g, but it is not maximal). So by recursion, we get an increasing function $f : \mathbf{On} \to a$. But that is a contradiction, because then by replacement we get that **On** is a set.
(3) implies (1): Let a be a set such that $\emptyset \notin a$, and we want a function $f : a \to \bigcup a$ such that $f(x) \in x$. Let b be the set of all functions $g : c \to \bigcup c$ such that $g(x) \in x$ where $c \subseteq a$. Define an ordering on b by $g_1 \leq g_2$ iff $g_1 \subseteq g_2$. Easily b satisfies the condition of (2) (it is not empty because $\emptyset \in b$), so there is a maximal element, f. We must show that the domain of f is a. If not, let $x \in a$, and so $x \neq \emptyset$, so there is some $y \in x$, and let $f' = f \cup \{(x, a)\}$. Then $f < f'$ – a contradiction.