Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

MODEL THEORY – EXERCISE 4

To be submitted on Wednesday 11.05.2011 by 14:00 in the mailbox.

Definition.

Suppose L is a signature.

(1) A *theory* for L is a set of sentences for L which is consistent (i.e. has a model) and deductively closed (i.e. closed under $\models$).
(2) A theory T is said to *eliminate quantifiers* if for every formula $\varphi(x_0, \ldots, x_{n-1})$ there exists a quantifier free formula $\psi(x_0, \ldots, x_{n-1})$ such that

$$T \models \forall x_0 \forall x_1 \ldots \forall x_{n-1} \varphi \leftrightarrow \psi.$$

(3) Given a structure M, let $Th(M) = \{\varphi \,|M \models \varphi\}$. This is *the theory of* M.
(4) For a set of sentences Σ, $Mod(\Sigma) = \{M \,|M \models \Sigma\}$.
(5) For a class of structures K, $Th(K) = \{\varphi \,|\forall M \in K\,(M \models \varphi)\}$.
(6) A class of structures K is called elementary if it is of the form $Mod(\Sigma)$ for a set of sentences Σ.
(7) A theory T is called complete if for every sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$.

Question 1.

Let $L = \{\in\}$ be the signature of set theory, i.e. $\in$ is a binary relation symbol and we write $x \in y$ instead of $\in(x, y)$. Of course, $\approx$ is also in L. Write down the following statements as formulas:

(1) $\varphi_\emptyset(x)$: "x is the empty set".
(2) $\varphi_\subseteq(x, y)$: "x is a subset of y".
(3) $\varphi_1(x)$: "x is a singleton, i.e. a set that contains only one element".
(4) $\varphi_2(x)$: "x is a pair, i.e. a set that contains exactly 2 elements".
(5) $\varphi_{op}(x, u, v)$: "x is the ordered pair (u, v), i.e. x is the set $\{\{u\}, \{u, v\}\}$".
(6) $\varphi_p(x)$: "x is an ordered pair, i.e. there are some u, v such that $x = (u, v)$".
(7) $\varphi_f(x, y, z)$: "x is a function from y to z" (i.e. x is a graph of a function from y to z).
(8) $\varphi_{\text{inj}}(x, y, z)$: "x is an injective function from y to z".
(9) $\varphi_{\text{onto}}(x, y, z)$: "x is a surjective function from y to z".
(10) $\varphi_\infty(x)$: "x is an infinite set" (hint: x is infinite iff there exists a function from a strict subset of x onto x).
(11) ψ : "there exists an infinite set ω such that for any infinite set x there exists an injective function from ω to x".
(12) What does the following formula say about x and y:

$$\forall z\,(z \in x \to \varphi_p(z) \wedge \forall r \forall w\,(\varphi_{op}(z, r, w) \to (r \in y \wedge w \in y))) \wedge$$
$$\forall z\,(z \in x \to \forall r \forall w\,(\varphi_{op}(z, r, w) \to \exists z'\,(\varphi_{op}(z', w, r) \wedge z' \in x))) \wedge$$
$$\forall z \forall z'\,((z \in x \wedge z' \in x) \to \forall r \forall w \forall u\,(\varphi_{op}(z, r, w) \wedge \varphi_{op}(z', w, u) \to \exists z''\,(\varphi_{op}(z'', r, u) \wedge z'' \in x))) \wedge$$
$$\forall z\,(z \in y \to \exists z'\,(z' \in x \wedge \varphi_{op}(z', z, z)))$$

Solution: x is an equivalence relation on y.

Question 2.
Let $L = \{<\}$. Let $M = (\mathbb{Q}, <)$ the structure of rational numbers with ordering. Given $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{Q}^n$, let the *order type* of $\bar{a}$ be the set of formulas

$$\{x_i < x_j \,|i, j < n, a_i < a_j\} \cup \{x_i \approx x_j \,|i, j < n, a_i = a_j\}.$$

Denote this by $\mathrm{otp}(\bar{a})$. Show that if $\bar{a}$ and $\bar{b}$ are two tuples of length n with the same order type, then there is an automorphism of M that takes $\bar{a}$ to $\bar{b}$.
Solution: use a piecewise linear map.

Question 3.
Show that the theory $Th(\mathbb{Q}, <)$ eliminates quantifiers. Work in the signature that has the additional logical connector $\perp$ which is always interpreted as false.
Use the following step.

1. (1) Explain why every sentences is equivalent to a quantifier free sentence.
   Solution: the theory is complete.
2. (2) Given a formula $\varphi(\bar{x})$ where $\bar{x} = (x_0, \ldots, x_{n-1})$ with $n > 0$, let $P = \{\mathrm{otp}(\bar{a}) \,|\mathbb{Q} \models \varphi[\bar{a}]\}$. Show that P is a finite set.
   Solution: for $i, j < n$ there are three options, $x_i < x_j \in \mathrm{otp}(\bar{a})$, $x_j < x_i \in \mathrm{otp}(\bar{a})$ or neither appear. Hence $|P| \leq \left(n^2\right)^3$.
3. (3) Show that $\varphi(\bar{x})$ is equivalent modulo $Th(\mathbb{Q}, <)$ to $\psi(\bar{x}) = \bigvee_{p \in P} \bigwedge p$ (where $\bigwedge p$ is the conjunction of all formulas in p) and conclude.
   Solution: if $\varphi(\bar{a})$ holds, then by definition $\mathrm{otp}(\bar{a}) \in p$, so $\bigwedge \mathrm{otp}(\bar{a})$ holds. If $\psi(\bar{a})$ holds, then necessarily $\mathrm{otp}(\bar{a}) \in P$, so there is some $\bar{b}$ with the same order type such that $\varphi\left(\bar{b}\right)$ holds. But then there is an automorphism of $\mathbb{Q}$ that takes $\bar{b}$ to $\bar{a}$, so $\varphi(\bar{a})$ also holds. Obviously ψ is quantifier free.

Question 4.
Let L be a signature. Let Σ, Σ_1, Σ_2 be set of sentences in L, and K, M non-empty classes of L-structures. Show that:

1. (1) $\Sigma_1 \subseteq \Sigma_2 \Rightarrow Mod(\Sigma_2) \subseteq Mod(\Sigma_1)$.
2. (2) $K \subseteq M \Rightarrow Th(M) \subseteq Th(K)$.
3. (3) $\Sigma \subseteq Th(Mod(\Sigma))$.
4. (4) $K \subseteq Mod(Th(K))$.
5. (5) $Mod(\Sigma) = Mod(Th(Mod(\Sigma)))$.
   Solution: use (1), (3) and (4). By (4) we get $\subseteq$, by (1) and (3) we get $\supseteq$.
6. (6) $Th(K) = Th(Mod(Th(K)))$.
   Solution: use (2), (3) and (4) as before.
7. (7) Show that Σ is a theory iff Σ is consistent and $\Sigma = Th(Mod(\Sigma))$.
8. (8) Show that K is elementary iff $K = Mod(Th(K))$.
   Solution: by (5).
9. (9) In class you showed that if $\Sigma = Th(A)$ for a structure A, then Σ is complete. Is it true that if $\Sigma = Th(\{A_1, A_2\})$ then it must be complete? Give a property of a class of structures K, such that $Th(K)$ is complete iff K satisfies this property.
   Solution: no. A_1 might have 2 elements and A_2 only 1. The property is: if $M, N \in K$ then $M \equiv N$ (meaning they have the same theory).