REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
(Vorlesung 29b - für 09/02/2023)

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Contents

1. APPLICATION OF SPSS TO THE MOMENT PROBLEM (continued)

Lemma 1.1. (Lemma 2.11 of last lecture) Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ be a linear functional and denote by

$$\tau : \mathbb{N}_0^n \to \mathbb{R}$$

the corresponding multisequence (i.e. $\tau(\underline{k}) := L(\underline{X}^{\underline{k}})\ \forall\ \underline{k} \in \mathbb{N}_0^n$).
Fix $g \in \mathbb{R}[\underline{X}]$. Then $L(h^2 g) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ if and only if the multisequence $g(E)_\tau$ is psd.

Proof. Compute:

1. $L(\underline{X}^{\underline{l}} g) = \sum\limits_{\underline{k}\, \in\, \mathbb{Z}_+{}^n} a_{\underline{k}} \tau(\underline{k} + \underline{l}) = g(E)_\tau(\underline{l})$; for all $\underline{l} \in \mathbb{N}_0^n$.

Thus if $h = \sum\limits_i c_i \underline{X}^{\underline{k}_i} \in \mathbb{R}[\underline{X}]$ then $h^2 = \sum\limits_{i,j} c_i c_j \underline{X}^{\underline{k}_i + \underline{k}_j}$.

2. So, $L(h^2 g) = L\Big[\big(\sum\limits_{i,j} c_i c_j \underline{X}^{\underline{k}_i + \underline{k}_j}\big) g\Big] = \sum\limits_{i,j} c_i c_j L(\underline{X}^{\underline{k}_i + \underline{k}_j} g)$

$$\underbrace{=}_{[\text{by } 1.]} \sum_{i,j} g(E)_\tau(\underline{k}_i + \underline{k}_j) c_i c_j. \qquad \square$$

Theorem 1.2. (Schmüdgen's NNSS) (Reformulation in terms of moment sequences) Let $K = K_S$ compact, $S = \{g_1, \ldots, g_s\}$ and $\tau : \mathbb{N}_0^n \to \mathbb{R}$ be a given multisequence. Then τ is a K-moment sequence if and only if the multisequences $(g_1^{e_1} \ldots g_s^{e_s})(E)_\tau : \mathbb{N}_0^n \to \mathbb{R}$ are all psd for all $(e_1, \ldots, e_s) \in \{0, 1\}^s$. □

Next we reformulate question (1) in 2.4 of Lecture 15 in terms of Hankel matrices and bilinear forms.

2. SCHMÜDGEN'S NNSS, HANKEL MATRICES AND BILINEAR FORMS

We want to understand $L(h^2 g) \geq 0; h, g \in \mathbb{R}[\underline{X}]$ in terms of Hankel matrices.

Definition 2.1. A real symmetric $n \times n$ matrix A is **psd** if $\underline{x}^T A \underline{x} \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n$.
An $\mathbb{N} \times \mathbb{N}$ symmetric matrix (say) A is psd if $\underline{x}^T A \underline{x} \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n$ and $\forall \ n \in \mathbb{N}$.

Definition 2.2. Let $L \neq 0; L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ be a given linear functional. Fix $g \in \mathbb{R}[\underline{X}]$. Consider symmetric bilinear form:

$$\begin{aligned} \langle \ , \ \rangle_g &: \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] \to \mathbb{R} \\ \langle h, k \rangle_g &:= L(hkg) \ ; \ h, k \in \mathbb{R}[\underline{X}] \end{aligned}$$

Denote by S_g the $\mathbb{N}\times\mathbb{N}$ real symmetric matrix with $\underline{\alpha}\underline{\beta}$-entry $\langle \underline{X}^{\underline{\alpha}}, \underline{X}^{\underline{\beta}} \rangle_g \ \forall \ \underline{\alpha}, \underline{\beta} \in \mathbb{N}_0^n$, i.e. the $\underline{\alpha}\underline{\beta}$-entry of S_g is $L(\underline{X}^{\underline{\alpha}+\underline{\beta}} \ g)$.

Example. Let $g = 1$, then

$\langle \underline{X}^{\underline{\alpha}}, \underline{X}^{\underline{\beta}} \rangle_1 = L(\underline{X}^{\underline{\alpha}+\underline{\beta}}) := s_{\underline{\alpha}+\underline{\beta}}$.

More generally, if $g = \sum a_{\underline{\gamma}} \ \underline{X}^{\underline{\gamma}}$ then

$$\langle \underline{X}^{\underline{\alpha}}, \underline{X}^{\underline{\beta}} \rangle_g = L\Big(\sum_{\gamma} a_{\underline{\gamma}} \ \underline{X}^{\underline{\alpha}+\underline{\beta}+\underline{\gamma}} \Big) = \sum_{\underline{\gamma}} a_{\underline{\gamma}} \ s_{\underline{\alpha}+\underline{\beta}+\underline{\gamma}} \ .$$

Proposition 2.3. Let L, g be fixed as above. Then the following are equivalent:

1. $L(\sigma g) \geq 0 \ \forall \ \sigma \in \sum \mathbb{R}[\underline{X}]^2$.
2. $L(h^2 g) \geq 0 \ \forall \ h \in \mathbb{R}[\underline{X}]$.
3. $\langle \ , \ \rangle_g$ is psd (i.e. $\langle h, h \rangle_g \geq 0$ for all $h \in \mathbb{R}[\underline{X}]^2$).
4. S_g is psd.

Proof. (1) $\Leftrightarrow$ (2) is clear.
Since $\langle h, h \rangle_g = L(h^2 g)$, (2) $\Leftrightarrow$ (3) is clear.
(3) $\Leftrightarrow$ (4) is also clear. □

2.4. Example. (Hamburger) Let $n = 1$. A linear functional $L : \mathbb{R}[X] \to \mathbb{R}$ comes from a Borel measure on $\mathbb{R}$ if and only if $L(\sigma) \geq 0$ for every $\sigma \in \sum \mathbb{R}[X]^2$.

Proof. From Haviland we know L comes from a Borel measure iff $L(f) \geq 0$ for every $f(X) \in \mathbb{R}[X], f \geq 0$ on $\mathbb{R}$. But $\mathrm{Psd}(\mathbb{R}) = \sum \mathbb{R}[X]^2$ (by exercise in Real Algebraic Geometry course in WS 2009-10). So the condition is clear. □

Remark 2.5. We can express Hamburgers's Theorem via Hankel matrix S_g with $g = 1$ the constant polynomial since $n = 1$, so (for $i, j \in \mathbb{N}$) the ij $^{\text{th}}$ coefficient of S_1 is $s_{i+j} = L(X^{i+j})$.

Hence, $S_1 = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & \dots & \\ & & \ddots & \\ s_2 & \dots & & \\ \dots & \dots & & \end{pmatrix}$ is psd.

END OF RAG I IN WISE 2022/2023

2.1. REFORMULATION OF SCHMÜDGEN'S SOLUTION TO THE MOMENT PROBLEM IN TERMS OF HANKEL MATRICES

2.6. Let $S = \{ g_1, \dots, g_s\} \subseteq \mathbb{R}[\underline{X}]$ and $K_S \subseteq \mathbb{R}^n$ is compact. A linear functional L on $\mathbb{R}[\underline{X}]$ is represented by a Borel measure on K iff the 2^S $\mathbb{N} \times \mathbb{N}$ Hankel matrices $\{S_{g_1^{e_1} \dots g_s^{e_s}} | (e_1, \dots, e_s) \in \{0, 1\}^s\}$ are psd, where $S_g := [L(\underline{X}^{\underline{\alpha}+\underline{\beta}} g)]_{\underline{\alpha},\underline{\beta}}$; $\underline{\alpha}, \underline{\beta} \in \mathbb{N}^n$.

3. FINITE SOLVABILITY OF THE K- MOMENT PROBLEM

Definition 3.1. Let K be a basic closed semi-algebraic subset of $\mathbb{R}^n$.

1. The K-moment problem (**KMP**) is **finitely solvable** if there exists S finite, $S \subseteq \mathbb{R}[\underline{X}]$ such that:

   (i) $K = K_S$, and

   (ii) $\forall$ linear functional L on $\mathbb{R}[\underline{X}]$ we have: $L(T_S) \geq 0 \Rightarrow L(\mathrm{Psd}(K)) \geq 0$ (equivalently, (iii) $L(T_S) \geq 0 \Rightarrow \exists\, \mu : L = \int d\mu$).

2. We shall say S **solves the KMP** if (i) and (ii) (equivalently (i) and (iii)) hold.

3.2. Schmüdgen's solution to the KPM for K compact b.c.s.a. Let $K \subseteq \mathbb{R}^n$ be a compact basic closed semi-algebraic set. Then S solves the KMP for any finite description S of K (i.e. for all finite $S \subseteq \mathbb{R}[\underline{X}]$ with $K = K_S$).

One can restate the condition "S solves the K-Moment problem" via the equality of two preorderings. We shall adopt this approach throughout:

Definition 3.3. Let $T_S \subseteq \mathbb{R}[\underline{X}]$ be a preordering. Define the **dual cone** of T_S:

$$T_S^{\vee} := \{L \mid L : \mathbb{R}[\underline{X}] \to \mathbb{R} \text{ is a linear functional}; L(T_S) \geq 0\},$$

and the **double dual cone**:

$$T_S^{\vee\vee} := \{f \mid f \in \mathbb{R}[\underline{X}]; L(f) \geq 0 \ \forall \ L \in T_S^{\vee}\}.$$

Lemma 3.4. For $S \subseteq \mathbb{R}[\underline{X}]$, S finite:

(a) $T_S \subseteq T_S^{\vee\vee}$

(b) $T_S^{\vee\vee} \subseteq \operatorname{Psd}(K_S)$.

Proof. (a) Immediate by definition.

(b) Let $f \in T_S^{\vee\vee}$. To show: $f(\underline{x}) \geq 0 \ \forall \ \underline{x} \in K_S$.

Now every $\underline{x} \in \mathbb{R}^n$ determines an $\mathbb{R}$-algebra homomorphism

$$e_{v_x} := L_{\underline{x}} \in \operatorname{Hom}(\mathbb{R}[\underline{X}], \mathbb{R}); \ L_{\underline{x}}(g) = e_{v_x}(g) := g(\underline{x}) \ \forall \ g \in \mathbb{R}[\underline{X}],$$

this $L_{\underline{x}}$ is in particular a linear functional.

Moreover we claim that $L_{\underline{x}}(T_S) \geq 0$ for $\underline{x} \in K_S$. Indeed if $g \in T_S$ then $L_{\underline{x}}(g) = g(\underline{x}) \geq 0$ for $\underline{x} \in K_S$.

So, by assumption on f we must also have $L_{\underline{x}}(f) \geq 0$ for $\underline{x} \in K_S$, i.e. $f(\underline{x}) \geq 0$ for all $\underline{x} \in K_S$ as required.

□

We summarize as follows:

Corollary 3.5. For finite $S \subseteq \mathbb{R}[\underline{X}]$:

$$T_S \subseteq T_S^{\vee\vee} \subseteq \operatorname{Psd}(K_S).$$

Corollary 3.6. (Reformulation of finite solvability) Let $K \subseteq \mathbb{R}^n$ be a b.c.s.a. set and $S \subseteq \mathbb{R}[\underline{X}]$ be finite. Then S solves the KMP iff

(j) $K = K_S$, and

(jj) $T_S^{\vee\vee} = \mathrm{Psd}(K)$.

Proof. Assume (ii) of definition 3.1, i.e. $\forall\, L : L(T_S) \geq 0 \Rightarrow L(\mathrm{Psd}(K)) \geq 0$, and show (jj) i.e. $T_S^{\vee\vee} = \mathrm{Psd}(K)$:
Let $f \in \mathrm{Psd}(K)$. Show $f \in T_S^{\vee\vee}$ i.e. show $L(f) \geq 0\ \forall\, L \in T_S^{\vee}$.
Assume $L(T_S) \geq 0$. Then by assumption $L(\mathrm{Psd}(K)) \geq 0$. So, $L(f) \geq 0$ as required.

Conversely, assume (jj) and show (ii):
Let $L(T_S) \geq 0$, i.e. $L \in T_S^{\vee}$. Show $L(\mathrm{Psd}(K)) \geq 0$, i.e show $L(f) \geq 0\ \forall\, f \in \mathrm{Psd}(K)$.
Now [by assumption (jj)] $f \in \mathrm{Psd}(K) \Rightarrow f \in T_S^{\vee\vee} \Rightarrow L(f) \geq 0\ \forall\, L \in T_S^{\vee}$. □

We shall come back later to $T_S^{\vee\vee}$ and describe it as closure w.r.t. an appropriate topology.

4. HAVILAND'S THEOREM

For the proof of Haviland's theorem (2.5 of lecture 15), we will recall Riesz Representation Theorem.

Definition 4.1. A topological space is said to be **Hausdorff** (or **seperated**) if it satisfies
(H4): any two distinct points have disjoint neighbourhoods, or
(T_2): two distinct points always lie in disjoint open sets.

Definition 4.2. A topological space χ is said to be **locally compact** if $\forall\, x \in \chi\ \exists$ an open neighbourhood $\mathcal{U} \ni x$ such that $\overline{\mathcal{U}}$ is compact.

Theorem 4.3. (Riesz Representation Theorem) Let χ be a locally compact Hausdorff space and $L : \mathrm{Cont}_c(\chi, \mathbb{R}) \to \mathbb{R}$ be a positive linear functional i.e. $L(f) \geq 0\ \forall\, f \geq 0$ on χ. Then there exists a unique (positive regular) Borel measure μ on χ such that $L(f) = \int_\chi f d\mu\ \ \forall\, f \in \mathrm{Cont}_c(\chi, \mathbb{R})$, where $\mathrm{Cont}_c(\chi, \mathbb{R}) :=$ the ring ($\mathbb{R}$-algebra) of all continuous functions $f : \chi \to \mathbb{R}$ (addition and multiplication defined pointwise) with compact support i.e. such that the set $\mathrm{supp}(f) := \{x \in \chi : f(x) \neq 0\}$ is compact.

Definition 4.4. L **positive** means:

$$L(f) \geq 0\ \forall\, f \in \mathrm{Cont}_C(\chi, \mathbb{R}) \text{ with } f \geq 0 \text{ on } \chi.$$