REAL ALGEBRAIC GEOMETRY LECTURE NOTES PART II: POSITIVE POLYNOMIALS (Vorlesung 29a - für 09/02/2023)

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1. SCHMÜDGEN'S NICHTNEGATIVSTELLENSATZ AND LINEAR FUNCTIONALS ON $\mathbb{R}[\underline{X}]$

1.1. Schmüdgen's Nichtnegativstellensatz : Let K_S be a compact basic closed semi algebraic set and $f \in \mathbb{R}[\underline{X}]$. Then

$$f \geq 0 \text{ on } K_S \Rightarrow \forall \epsilon \text{ real}, \epsilon > 0 : f + \epsilon \in T_S.$$

Corollary 1.2. Let $K = K_S$ be a compact basic closed semi algebraic set and $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ be a linear functional with $L(1) = 1$. Then

$$\underbrace{L(T_S) \geq 0}_{(\text{i.e. } L(f) \geq 0 \ \forall f \in T_S)} \quad \Rightarrow \quad \underbrace{L(\mathrm{Psd}(K_S)) \geq 0}_{(\text{i.e. } L(f) \geq 0 \ \forall f \geq 0 \text{ on } K_S).}$$

Proof. Let $f \in \mathrm{Psd}(K_S)$ and assume $L(T_S) \geq 0$,

To show: $L(f) \geq 0$

By 1.1, $\forall \epsilon > 0 : f + \epsilon \in T_S$

So, $L(f + \epsilon) \geq 0$ i.e. $L(f) \geq -\epsilon \ \forall \epsilon > 0$ real

$\Rightarrow L(f) \geq 0.$ □

We shall now relate this to the **problem of representation** of linear functionals via integration along measures (i.e. $\int d\mu$).

2. APPLICATION OF SPSS TO THE MOMENT PROBLEM

Let $\mathcal{X}$ be a locally compact Hausdorff topological space.

Definition 2.1. $\mathcal{X}$ is **locally compact** if $\forall\, x \in \mathcal{X}\ \exists$ open $\mathcal{U}$ in $\mathcal{X}$ s.t. $x \in \mathcal{U}$ and $\overline{\mathcal{U}}$ (closure) is compact.

Notation 2.2. $\mathcal{B}^\delta(\mathcal{X})$:= set of Borel measurable sets in $\mathcal{X}$
= the smallest family of subsets of $\mathcal{X}$ containing all compact subsets of $\mathcal{X}$, closed under finite $\bigcup$, set theoretic difference $A \setminus B$ and countable $\bigcap$.

Definition 2.3. A **Borel measure** μ on $\mathcal{X}$ is a positive measure on $\mathcal{X}$ s.t. every set in $\mathcal{B}^\delta(\mathcal{X})$ is measurable. We also require our measure to be **regular** i.e. $\forall B \in \mathcal{B}^\delta(\mathcal{X})$ and $\forall\ \epsilon > 0\ \exists\ K, \mathcal{U} \in \mathcal{B}^\delta(\mathcal{X}), K$ compact, $\mathcal{U}$ open s.t. $K \subseteq B \subseteq \mathcal{U}$ and $\mu(K) + \epsilon \geq \mu(B) \geq \mu(\mathcal{U}) - \epsilon$.

2.4. Moment problem is the following:
Given a closed set $K \subseteq \mathbb{R}^n$ and a linear functional $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$
Question:

$$\text{when does } \exists \text{ a Borel measure } \mu \text{ on } K \text{ s.t. } \forall\, f \in \mathbb{R}[\underline{X}] : L(f) = \int f d\mu\ ? \tag{1}$$

Necessary condition for (1): $\forall\, f \in \mathbb{R}[\underline{X}], f \geq 0$ on $K \Rightarrow L(f) \geq 0$ (2)

in other words: $L(\mathrm{Psd}(K)) \geq 0$ (3)

Is this necessary condition also sufficient?
The answer is YES.

Theorem 2.5. (Haviland) Given $K \subseteq \mathbb{R}^n$ closed and $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ a linear functional with $L(1) = 1$:

$$\exists\, \mu \text{ as in (1) iff } \forall\, f \in \mathbb{R}[\underline{X}] : L(f) \geq 0 \text{ if } f \geq 0 \text{ on } K.$$

We shall prove Haviland's Theorem later. For now we shall deduce a corollary to SPSS.

Corollary 2.6. Let $K_S = \{\underline{x} \mid g_i(\underline{x}) \geq 0; i = 1, \ldots, s\} \subseteq \mathbb{R}^n$ be a basic closed semi-algebraic set and compact, $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$ a linear functional with $L(1) = 1$. If $L(T_S) \geq 0$, then $\exists\, \mu$ positive Borel measure on K s.t. $L(f) = \int_{K_S} f d\mu\ \ \forall\, f \in \mathbb{R}[\underline{X}]$.

Remark 2.7. Let $S = \{g_1, \ldots, g_s\}$.
1. $L(T_S) \geq 0$ can be written as

$$L(h^2 g_1^{e_1} \ldots g_s^{e_s}) \geq 0 \ \forall\ h \in \mathbb{R}[\underline{X}], e_1, \ldots, e_s \in \{0, 1\}.$$

2. Compare Haviland to Schmüdgen's moment problem, for compact K_S: we do not need to check $L(\mathrm{Psd}(K_S)) \geq 0$ we only need to check $L(T_S) \geq 0$.

3. Reformulation of question (1) (in 2.4) in terms of moment sequences:
Let $L : \mathbb{R}[\underline{X}] \longrightarrow \mathbb{R}$, with $L(1) = 1$. Consider $\{\underline{X}^{\underline{\alpha}} = X_1^{\alpha_1} \ldots X_n^{\alpha_n};\ \underline{\alpha} \in \mathbb{N}_0^n\}$ a monomial basis for $\mathbb{R}[\underline{X}]$. So L is completely determined by the (multi)sequence of real numbers $\tau(\underline{\alpha}) := L(\underline{X}^{\underline{\alpha}})$; $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, i.e. by the function $\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$ is a function) and conversely, every such sequence determines a linear functional L :

$$L\Big(\sum_{\underline{\alpha}} a_{\underline{\alpha}} \underline{X}^{\underline{\alpha}}\Big) := \sum_{\underline{\alpha}} a_{\underline{\alpha}} \tau(\underline{\alpha})).$$

So, (1) (in 2.4) can be reformulated as:
Given $K \subseteq \mathbb{R}^n$ closed, and a multisequence $\tau = \tau(\underline{\alpha})_{\underline{\alpha} \in \mathbb{N}_0^n}$ of real numbers, $\exists\ \mu$ positive Borel measure on K s.t $\int_K \underline{X}^{\underline{\alpha}}\, d\mu = \tau(\underline{\alpha})$ for all $\underline{\alpha} \in \mathbb{N}_0^n$?

Definition 2.8. A function $\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$ is a $K-$**moment sequence** if $\exists\ \mu$ positive borel measure on K s.t $\tau(\underline{\alpha}) = \displaystyle\int_K \underline{X}^{\underline{\alpha}}\, d\mu$ for all $\underline{\alpha} \in \mathbb{N}_0^n$

So (1) can be reformulated as: given K and a function $\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$, when is τ a $K-$moment sequence?

Definition 2.9. A function $\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$ is called **psd** if

$$\sum_{i,j=1}^{m} \tau\,(\underline{k}_i + \underline{k}_j)\, c_i\, c_j \geq 0,$$

for $m \geq 1$, arbitrary distinct $\underline{k}_1, \ldots, \underline{k}_m \in \mathbb{N}_0^n; c_1, \ldots, c_m \in \mathbb{R}$.

Definition 2.10. Given $\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$ a function and a fixed polynomial $g(\underline{X}) = \displaystyle\sum_{\underline{k}\, \in\, \mathbb{N}_0{}^n} a_{\underline{k}}\ \underline{X}^{\underline{k}} \in \mathbb{R}[\underline{X}]$. Define a new function $g(E)_\tau : \mathbb{N}_0^n \longrightarrow \mathbb{R}$ by $g(E)_\tau(\underline{l}) := \displaystyle\sum_{\underline{k}\, \in\, \mathbb{N}_0{}^n} a_{\underline{k}}\ \tau(\underline{k} + \underline{l})$; for any $\underline{l} \in \mathbb{N}_0^n$.

Lemma 2.11. Let $L : \mathbb{R}[\underline{X}] \to \mathbb{R}$ be a linear functional and denote by

$$\tau : (\mathbb{N}_0)^n \to \mathbb{R}$$

the corresponding multisequence (i.e. $\tau(\underline{k}) := L(\underline{X}^{\underline{k}})\ \forall\ \underline{k} \in (\mathbb{N}_0)^n$).
Fix $g \in \mathbb{R}[\underline{X}]$. Then $L(h^2 g) \geq 0$ for all $h \in \mathbb{R}[\underline{X}]$ if and only if the multisequence $g(E)_\tau$ is psd.