REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
(Vorlesung 28 - für 07/02/2023)

SALMA KUHLMANN

Contents

1. Rings of bounded elements

Let A be a commutative ring with 1, $\mathbb{Q} \subseteq A$ and M be a quadratic module $\subseteq A$.

Definition 1.1. Consider

$$B_M = \{a \in A \mid \exists\, n \in \mathbb{N} \text{ s.t. } n + a \text{ and } n - a \in M\},$$

B_M is called the **ring of bounded elements**, which are bounded by M.

Proposition 1.2.

(1) M is an archimedean module of A iff $B_M = A$.

(2) B_M is a subring of A.

(3) $\forall\, a \in A$, $a^2 \in B_M \Rightarrow a \in B_M$.

(4) More generally, $\forall\, a_1, \ldots, a_k \in A$, $\displaystyle\sum_{i=1}^{k} a_i^2 \in B_M \Rightarrow a_i \in B_M\ \forall\, i = 1, \ldots, k$.

Proof. (1) Clear.

(2) Clearly $\mathbb{Q} \subseteq B_M$ and B_M is an additive subgroup of A.

To show: $a, b \in B_M \Rightarrow ab \in B_M$
Using the identity

$$ab = \frac{1}{4}\big[(a+b)^2 - (a-b)^2\big],$$

we see that in order to show that B_M is closed under multiplication it is sufficient to show that: $\forall\, a \in A: \ a \in B_M \Rightarrow a^2 \in B_M$.

Let $a \in B_M$. Then $n \pm a \in M$ for some $n \in \mathbb{N}$. Now $n^2 + a^2 \in M$.
Also $2n(n^2 - a^2) = (n^2 - a^2)[(n+a) + (n-a)]$.

$$\text{So, } (n^2 - a^2) = \frac{1}{2n}\Big[(n+a)(n^2-a^2) + (n-a)(n^2-a^2)\Big]$$
$$= \frac{1}{2n}\Big[(n+a)^2(n-a) + (n-a)^2(n+a)\Big] \in M.$$

So $(n^2 + a^2)$ and $(n^2 - a^2)$ both $\in M$. So by definition $a^2 \in B_M$. □ (2)

(3) Assume $a^2 \in B_M$. Say $n - a^2 \in M$, for some $n \in \mathbb{N}$, then use the identity:

$$(n \pm a) = \frac{1}{2}\big[(n-1) + (n-a^2) + (a \pm 1)^2\big] \in M.$$

So, $a \in B_M$. □ (3)

(4) If $\sum a_j^2 \in B_M$. Say $\big(n - \sum a_j^2\big) \in M$, then for all i, we have

$$(n - a_i^2) = \Big(n - \sum a_j^2\Big) + \sum_{j \neq i} a_j^2 \in M.$$

So, $a_i^2 \in B_M$ and so by (3), $a_i \in B_M$. □ (4)
□

Corollary 1.3. Let M be a quadratic module of $\mathbb{R}[\underline{X}]$. Then M is archimedean iff there exists $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^{n} X_i^2 \in M$$

Proof. ($\Rightarrow$) Clear by definition of archimedeanness.
($\Leftarrow$) First note that $\mathbb{R}_+ \subseteq M$ so, $\mathbb{R} \subseteq B_M$ (B_M subring).
Also $N - \sum_{i=1}^{n} X_i^2$ and $N + \sum_{i=1}^{n} X_i^2 \in M$. Therefore by definition $\sum_{i=1}^{n} X_i^2 \in B_M$.
So (by Proposition 1.2) $X_1, \ldots, X_n \in B_M$. This implies $\mathbb{R}[X_1, \ldots, X_n] = B_M$ and so M is archimedean. □

2. SCHMÜDGEN'S POSITIVSTELLENSATZ

Theorem 2.1. Let $S = \{g_1, \dots g_s\} \subseteq \mathbb{R}[\underline{X}]$. Assume that $K := K_S = \{\underline{x} \mid g_i(\underline{x}) \geq 0\}$ is compact. Then there exists $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^{n} X_i^2 \in T_S =: T.$$

In particular T is an archimedean preordering (by Corollary 1.3) and thus $\forall f \in \mathbb{R}[\underline{X}]$: $f > 0$ on $K_S \Rightarrow f \in T$.

Proof. [Reference: Dissertation, Thorsten Wörmann]

- K compact $\Rightarrow K$ bounded $\Rightarrow \exists\, k \in \mathbb{N}$ such that $\Big(k - \sum_{i=1}^{n} X_i^2\Big) > 0$ on K.
- By applying the Positivstellensatz to above we get: $\exists\, p, q \in T$ such that $p\Big(k - \sum_{i=1}^{n} X_i^2\Big) = 1 + q$. So, $p\Big(k - \sum_{i=1}^{n} X_i^2\Big)^2 = (1+q)\big(k - \sum_{i=1}^{n} X_i^2\big)$. So, $(1+q)\Big(k - \sum_{i=1}^{n} X_i^2\Big) \in T$.
- Set $T' = T + \Big(k - \sum_{i=1}^{n} X_i^2\Big)T$. By Corollary 1.3, T' is an archimedean preordering. Therefore $\exists\, m \in \mathbb{N}$ such that $(m - q) \in T'$; say: $m - q = t_1 + t_2\Big(k - \sum_{i=1}^{n} X_i^2\Big)$ for some $t_1, t_2 \in T$.
- So, $(m-q)(1+q) = t_1(1+q) + t_2\Big(k - \sum_{i=1}^{n} X_i^2\Big)(1+q) \in T$. So $(m - q)(1+q) \in T$.
- Adding

$$(m-q)(1+q) = mq - q^2 + m - q \in T, \tag{1}$$

$$\Big(\frac{m}{2} - q\Big)^2 = \frac{m^2}{4} + q^2 - mq \in T. \tag{2}$$

yields

$$\Big(m + \frac{m^2}{4} - q\Big) \in T. \tag{3}$$

- Multiplying (3) by $k \in T$, and adding $\Big(k - \sum_{i=1}^{n} X_i^2\Big)(1+q) \in T$ and $q\Big(\sum_{i=1}^{n} X_i^2\Big) \in T$, yields

  $k\Big(m + \frac{m^2}{4} - q\Big) + \Big(k - \sum_{i=1}^{n} X_i^2\Big)(1+q) + q\Big(\sum_{i=1}^{n} X_i^2\Big) \in T$

  i.e. $km + k\frac{m^2}{4} + k - \sum_{i=1}^{n} X_i^2 \in T$

  i.e. $k\left(\frac{m}{2}+1\right)^2 - \sum_{i=1}^{n} X_i^2 \in T$

  Set $N := k\left(\frac{m}{2}+1\right)^2$. □

2.2. Final Remarks on Schmüdgen's Positivstellensatz (SPSS):

1. **Corollary (Schmüdgen's Nichtnegativstellensatz):**
   Let K_S be compact, $f \geq 0$ on $K_S \Rightarrow \forall \epsilon$ real, $\epsilon > 0 : f + \epsilon \in T_S$.

2. SPSS fails in general if we drop the assumption that "K is compact".
   For example:
   (i) Consider $n = 1,\ S = \{X^3\}$, then $K_S = [0, \infty)$ (noncompact). Take $f = X + 1$. Then $f > 0$ on K_S. Claim: $f \notin T_S$, indeed elements of T_S have the form $t_0 + t_1X^3$, where $t_0, t_1 \in \sum \mathbb{R}[X]^2$. We have shown before in Lecture 15, Example 2.4(1)(iii) that non zero elements of this preordering either have even degree or odd degree ≥ 3.
   (ii) Consider $n \geq 2, S = \emptyset$, then $K_S = \mathbb{R}^n$. Take strictly positive versions of the Motzkin polynomial
   $$m(X_1, X_2) := 1 - X_1^2X_2^2 + X_1^2X_2^4 + X_1^4X_2^2,$$
   i.e. $m_\epsilon := m(X_1, X_2) + \epsilon$; $\epsilon \in \mathbb{R}_+$. Then $m_\epsilon > 0$ on $K_S = \mathbb{R}^2$, and it is easy to show that $m_\epsilon \notin T_S = \sum \mathbb{R}[\underline{X}]^2\ \forall \epsilon \in \mathbb{R}_+$.

3. SPSS fails in general for a quadratic module instead of a preordering. [Mihai Putinar's question answered by Jacobi + Prestel in Dissertation of T. Jacobi (Konstanz)]

4. SPSS fails in general if the condition "$f > 0$ on K_S" is replaced by "$f \geq 0$ on K_S".

   Example (Stengle): Consider $n = 1, S = \{(1 - X^2)^3\}$, $K_S = [-1, 1]$ compact. Take $f := 1 - X^2 \geq 0$ on K_S but $1 - X^2 \notin T_S$. (This example has already been considered in Lecture 15, Example 2.4(1)(ii).

5. PSS holds for any real closed field but not SPSS:

   Example: Let R be a non archimedean real closed field. Take $n = 1, S = \{(1 - X^2)^3\}$, then $K_S = [-1, 1]_R = \{x \in R \mid -1 \leq x \leq 1\}$. Take $f = 1 + t - X^2$, where $t \in R^{>0}$ is an infinitesimal element (i.e. $0 < t < \epsilon$, for every positive rational ϵ). Then $f > 0$ on K_S. We claim that $f \notin T_S$:

   Let v be the natural valuation on R. So $v(t) > 0$. Now suppose for a contradiction that $f \in T_S$. Then

   $$1 + t - X^2 = t_0 + t_1(1 - X^2)^3;\ t_0, t_1 \in \sum R[X]^2 \qquad (1)$$

   Let $t_i = \sum f_{ij}^2$; for $i = 0, 1$ and $f_{ij} \in R[X]$.

   Let $s \in R$ be the coefficient of the lowest value appearing in the f_{ij}, i.e. $v(s) = \min\{v(a) \mid a \text{ is coefficient of some } f_{ij}\}$.

   Case I. if $v(s) \geq 0$, then applying the residue map $\left(\theta_v \longrightarrow \overline{R} := \frac{\overline{\theta_v}}{\mathcal{I}_v};\right.$ defined by $x \longmapsto \overline{x}$, where θ_v is the valuation ring and $\mathcal{I}_v$ is the valuation ideal $\left.\right)$ to (1), we obtain

   $$1 - X^2 = \overline{t_0} + \overline{t_1}(1 - X^2)^3$$

   and since $\overline{t_i} = \sum \overline{f_{ij}}^2 \in \sum \mathbb{R}[X]^2; i = 0, 1;$ we get a contradiction to Example 2.4(1)(ii) of Lecture 15.

   Case II. if $v(s) < 0$. Dividing f by s^2 and applying the residue map we obtain

   $$0 = \overline{\frac{t_0}{s^2}} + \overline{\frac{t_1}{s^2}}(1 - X^2)^3$$

   (Note that $v(s^2) = 2v(s)$ is $\min\{v(a) \mid a \text{ is coefficient of some } f_{ij}^2\}$, i.e. $v(s^2) \leq v(a)$ for any such coefficient a, so $\frac{f_{ij}^2}{s^2}$ has coefficients with value ≥ 0.)

   So we obtain

   $$0 = t_0' + t_1'(1 - X^2)^3, \text{ with } t_0', t_1' \in \sum \mathbb{R}[X]^2 \text{ not both zero.}$$

Since t_0', t_1' have only finitely many common roots in $\mathbb{R}$ and $1 - X^2 > 0$ on the finite set $(-1, 1)$, this is impossible. $\square$(claim)

6. SPSS holds over archimedean real closed fields.