REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. ARCHIMEDEAN MODULES

Let A be a commutative ring, $\mathbb{Q} \subseteq A$, T a preprime.

Definition 1.1. Let M a T-module. M is **archimedean** if:

$$\forall\ a \in A, \exists\ N \geq 1, N \in \mathbb{Z}_+ \text{ s.t. } N + a, N - a \in M\ .$$

Proposition 1.2. Let T be a generating preprime, M a maximal proper T-module. Assume that M is archimedean. Then $\exists$ a uniquely determined $\alpha \in \text{Hom}(A, \mathbb{R})$ s.t. $M = \alpha^{-1}\big(\mathbb{R}_+\big) = P_\alpha$.
(In particular, M is an ordering, not just a semi-ordering.)

Proof. Let $a \in A$, define: cut $(a) = \big\{r \in \mathbb{Q} \mid r - a \in M\big\}$, this is an **upper cut** in $\mathbb{Q}$ (i.e. final segment of $\mathbb{Q}$) .

Claim 1: $\text{cut}(a) \neq \emptyset$ and $\mathbb{Q}\backslash\big(\text{cut}(a)\big) := \text{L}(a) \neq \emptyset$, ($\text{L}(a)$ is a **lower cut** in $\mathbb{Q}$).

Proof of claim 1. Since M is archimedean $\exists\ n \geq 1$ s.t. $n - a \in M$, so $\text{cut}(a) \neq \emptyset$. Also $\exists\ m \geq 1$ s.t. $(m + a) \in M$.

If $-(m+1)-a \in M$, then adding we get $-1 \in M$, a contradiction (since M is proper). So we have $-(m+1)-a \notin M$, which implies that $-(m+1) \in L(a)$.

$\square$(claim 1)

Now define a map $\alpha : A \longrightarrow \mathbb{R}$ by

$$\alpha(a) := \inf\big(\text{cut}(a)\big)$$

α is well-defined by Claim 1.

Claim 2: $\alpha(1) = 1,\ \alpha(M) \subseteq \mathbb{R}_+;\ \alpha(a \pm b) = \alpha(a) \pm \alpha(b)\ \forall\ a, b \in A$ and $\alpha(tb) = \alpha(t)\ \alpha(b)\ \forall\ t \in T, b \in A$.

This is left as an exercise.

Claim 3: $\alpha(ab) = \alpha(a)\ \alpha(b)\ \forall\ a, b \in A$

Proof of claim 3. T generating $\Rightarrow a = t_1 - t_2,\ t_1, t_2 \in T$

so, $\alpha(ab) = \alpha(t_1 b - t_2 b) = \alpha(t_1 b) - \alpha(t_2 b)$

$$= \alpha(t_1)\alpha(b) - \alpha(t_2)\alpha(b) \text{ [by claim 2]}$$

$$= \big(\alpha(t_1) - \alpha(t_2)\big)\alpha(b) = \alpha(t_1 - t_2)\alpha(b) = \alpha(a)\alpha(b)\ .$$

□(claim 3)

Claim 4: $\alpha^{-1}\big(\mathbb{R}_+\big) = M$

Proof of claim 4. By Claim 2, $M \subseteq \alpha^{-1}\big(\mathbb{R}_+\big)$ so, by maximality of M and since $P_\alpha = \alpha^{-1}\big(\mathbb{R}_+\big)$ is an ordering it follows that $M = \alpha^{-1}\big(\mathbb{R}_+\big)$. □

Corollary 1.3. Let A be a commutative ring with $\mathbb{Q} \subseteq A$, T an archimedean preprime, M a proper T-module. Then $\chi_M \neq \emptyset$.

Proof. Since T is archimedean, T is generating (because $a = (n + a) - n$, for $a \in A$) and M is a proper archimedean module (archimedean module because for an archimedean preprime T, every T-module is also archimedean). By Zorn's lemma extend M to a maximal proper archimedean T-module Q. Apply Proposition 1.2 to Q to get $\alpha \in \text{Hom}(A, \mathbb{R})$ such that $Q = \alpha^{-1}\big(\mathbb{R}_+\big)$. This implies $M \subseteq \alpha^{-1}\big(\mathbb{R}_+\big)$. So, $\alpha \in \chi_M$, which implies $\Rightarrow$ $\chi_M \neq \emptyset$. □

2. Representation theorem (Stone-Krivine, Kadison-Dubois)

The following corollary (to Proposition 1.2 and Corollary 1.3) answers the question raised in the last lecture:

Corollary 2.1. (Stone-Krivine, Kadison-Dubois) Let A be a commutative ring with $\mathbb{Q} \subseteq A$, T an archimedean preprime in A, M a proper T-module. Let $a \in A$ and

$$\hat{a} : \chi \to \mathbb{R} \quad \text{defined by}$$

$$\hat{a}(\alpha) := \alpha(a)$$

If $\hat{a} > 0$ on χ_M, then $a \in M$.

Proof. Assume $\hat{a} > 0$ on χ_M, i.e. $\hat{a}(\alpha) > 0 \ \forall \ \alpha \in \chi_M$.
To show: $a \in M$

- Consider $M_1 := M - aT$

  Since $\alpha(a) > 0 \ \forall \ \alpha \in \chi_M$, we have $\chi_{M_1} = \emptyset$ [because if $\alpha \in \chi_{M_1}$, then $\alpha(M_1) \subseteq \mathbb{R}_+$. So, $\alpha(-a) = -\alpha(a) \geq 0$. So, $\alpha(a) \leq 0$, but $\alpha \in \chi_M$ so $\alpha(a) > 0$, a contradiction].

  So (since M_1 is an archimedean T-module), we can apply Corollary 1.3 to M_1 to deduce that $-1 \in M_1$.

  Write $-1 = s - at$, $s \in M, t \in T$

  $\Rightarrow \ at - 1 = s \in M$ $(\star)$

- Consider $\sum := \{r \in \mathbb{Q} \mid r + a \in M\}$

  We **claim** that: $\exists \ \rho \in \sum; \ \rho < 0$

  Once the claim is established we are done (with the proof of corollary) because

  $a = \underbrace{(a+\rho)}_{\in M} + \underbrace{(-\rho)}_{\in M} \in M$.

  Proof of the claim: First observe that $\sum \neq \emptyset$ $\big($since $\exists \ n \geq 1$ s.t $n+a \in T \subseteq M$, so $n \in \sum$ $\big)$.

  Now fix $r \in \sum$, $r \geq 0$ and fix an integer $k \geq 1$ s.t $(k-t) \in T$

  Write: $kr - 1 + ka = \underbrace{(k-t)}_{\in T}\underbrace{(r+a)}_{\in M} + \underbrace{(at-1)}_{\in M} + \underbrace{rt}_{\in M} \in M$ by $(\star)$.

  Multiplying by $\dfrac{1}{k}$, we get

  $\left(r - \dfrac{1}{k}\right) + a \in M$, i.e. $\left(r - \dfrac{1}{k}\right) \in \sum$

  Repeating we eventually find $\rho \in \sum$, $\rho < 0$. □

Notation 2.2. For a quadratic module $M \subseteq \mathbb{R}[\underline{X}]$, set

$$K_M := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \ \forall \ g \in M\}.$$

Note that if $M = M_S$ with $S = \{g_1, \dots, g_s\}$, then $K_S = K_M$.

We have the following corollaries to Corollary 2.1. (Stone-Krivine, Kadison-Dubois):

Corollary 2.3. (Putinar's Archimedean Positivstellensatz) Let $M \subseteq \mathbb{R}[\underline{X}]$ be an archimedean quadratic module. Then for each $f \in \mathbb{R}[\underline{X}]$:

$$f > 0 \text{ on } K_M \Rightarrow f \in M \ .$$

Corollary 2.4. Let $A = \mathbb{R}[\underline{X}]$ and $S = \{g_1, \dots, g_s\}$. Assume that the finitely generated preordering T_S is archimedean. Then for all $f \in A$:

$$f > 0 \ \text{ on } K_S \Rightarrow f \in T_S.$$

Remark 2.5.

1. To apply the corollary we need a criterion to determine when a preordering (quadratic module) is archimedean.
2. T_S is archimedean $\Rightarrow$ for $f = \sum X_i^2 : \exists\, N$ s.t. $N - f = N - \sum X_i^2 \in T_S$

   $\Rightarrow N - \sum X_i^2 \geq 0$ on K_S.

   $\Rightarrow K_S$ is bounded. Also K_S is closed.

   So T_S is archimedean implies K_S is compact.