REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. SCHMÜDGEN'S POSITIVSTELLENSATZ

Theorem 1.1. Let $S = \{g_1, \dots, g_s\}$ be a finite subset of $\mathbb{R}[X_1, \dots, X_n]$ and $K_S \subseteq \mathbb{R}^n$ be a compact basic closed semi algebraic set. And let $f \in \mathbb{R}[\underline{X}]$ s.t. $f > 0$ on K_S. Then $f \in T_S$.

To prove this we first need the Representation Theorem:

2. REPRESENTATION THEOREM (STONE-KRIVINE, KADISON-DUBOIS)

Let A be a commutative ring with 1. Let

$$\chi := \mathrm{Hom}(A, \mathbb{R}) = \big\{\alpha \mid \alpha : A \to \mathbb{R}, \alpha \text{ ring homomorphism}\big\}.$$

Notation 2.1. If $M \subseteq A$ denote

$$\chi_M = \big\{\alpha \in \chi \mid \alpha(M) \subseteq \mathbb{R}_+\big\} \ .$$

Notation 2.2. For $a \in A$ define a map

$$\hat{a} : \chi \to \mathbb{R} \quad \text{by}$$
$$\hat{a}(\alpha) := \alpha(a)$$

Remark 2.3. Let $M \subseteq A$, with notations 2.1 and 2.2 we see that

$$\begin{aligned} \chi_M &:= \big\{\alpha \in \chi |\ \alpha(M) \subseteq \mathbb{R}_+\big\} \\ &= \big\{\alpha \in \chi |\ \alpha(a) \geq 0, \forall\ a \in M\big\} \\ &= \big\{\alpha \in \chi |\ \hat{a}(\alpha) \geq 0, \forall\ a \in M\big\} \end{aligned}$$

So, χ_M is "the nonnegativity set" of M in χ.

Observation 2.4. $a \in M \Rightarrow \hat{a} \geq 0$ on χ_M, because if $\alpha \in \chi_M$, then $\hat{a}(\alpha) \geq 0$ (by definition).

Conversely, answer the question: for $a \in A$, if $\hat{a} > 0$ on $\chi_M \Rightarrow a \in M$?

Exkurs 2.5. One can view $\chi = \mathrm{Hom}(A, \mathbb{R})$ as a topological subspace of $(\mathrm{Sper}(A)$, spectral topology) as follows:

1. Embedding of $\mathrm{Hom}(A, \mathbb{R})$ in $\mathrm{Sper}(A)$:

   Consider the map defined by

   $$\mathrm{Hom}(A, \mathbb{R}) \to \mathrm{Sper}(A)$$

   $$\alpha \mapsto P_\alpha := \alpha^{-1}\big(\mathbb{R}_+\big),$$

   where (recall that) $\mathrm{Sper}(A) := \big\{P\ ;\ P \text{ is an ordering of } A\ \big\}$.

   Then (i) this map is well defined i.e. $P_\alpha \subseteq A$ is an ordering.

   (ii) this map is injective : $\alpha \neq \beta \Rightarrow P_\alpha \neq P_\beta$.

   (iii) $\mathrm{support}(P_\alpha) = \ker \alpha$.

2. Topology on χ :

   Endow χ with a topology : for $a \in A$

   $$U(\hat{a}) = \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}$$

   is a subbasis of open sets. Then

   (iv) for $a \in A$, the map $\hat{a} : \chi \to \mathbb{R}$ is continuous in this topology.

   (v) in fact this topology on χ is the weakest topology on χ for which $\hat{a}$ is continuous for all $a \in A$, i.e. if τ is any other topology on χ which makes all these maps $\hat{a}$ (for $a \in A$) continuous then τ has more open sets than this weakest topology (i.e. $U(\hat{a})$ lies in τ).

(vi) this topology is also the topology induced on χ via the embedding $\alpha \mapsto P_\alpha$ giving Sper(A) the spectral topology [just use the fact that $\hat{a}(\alpha) > 0 \Leftrightarrow a \notin -P_\alpha \Leftrightarrow a >_{P_\alpha} 0$. Spectral topology: $U(a) = \{P\ ;\ a \notin -P\} = \{P \mid a >_P 0\}$].

Now we are back to the question (in Observation 2.4): for $a \in A$, does $\hat{a} > 0$ on $\chi_M \Rightarrow a \in M$?
Yes under additional assumptions on the subset M that we shall now study:

3. PREPRIMES, MODULES AND SEMI-ORDERINGS IN RINGS

Let A be a commutative ring with 1 and $\mathbb{Q} \subseteq A$. The concept of preordering generalizes in two directions:

(i) Preprimes

(ii) Modules (special case: quadratic modules)

Definitions 3.1. (1) A **preprime** is a subset T of A such that

$$T + T \subseteq T; \quad TT \subseteq T; \quad \mathbb{Q}_+ \subseteq T.$$

(2) Let T be a preprime of A. $M \subseteq A$ is a T**-module** if

$$M + M \subseteq M; \quad TM \subseteq M; \quad 1 \in M \text{ (i.e. } T \subseteq M).$$

[Note that in particular, a preprime T is a T-module.]

(3) A preprime T of A is said to be **generating** if $T - T = A$.

[Note that if T is any preprime then $T - T$ is already a subring of A because

$(t_1 - t_2) + (t_3 - t_4) = (t_1 + t_3) - (t_2 + t_4)$

$(t_1 - t_2)(t_3 - t_4) = (t_1t_3 + t_2t_4) - (t_1t_4 + t_2t_3)$.]

Proposition 3.2. Every preordering T of A is a generating preprime.

Proof. (i) For $\frac{m}{n} \in \mathbb{Q} : \frac{m}{n} = \left(\frac{1}{n}\right)^2 mn = \underbrace{\frac{1}{n^2} + \ldots + \frac{1}{n^2}}_{\text{(mn-times)}}$

so $\mathbb{Q}_+ \subseteq T$.

(ii) For $a \in A$, $a = \left(\frac{1+a}{2}\right)^2 - \left(\frac{1-a}{2}\right)^2$.

So $A = T - T$. □

Definitions 3.3. (1) A **quadratic module** is a T-module over the preprime $T = \sum A^2$.

(2) A T-module M is **proper** if $(-1) \notin M$.

(3) A **semi-ordering** M is a quadratic module such that moreover

$$M \cup (-M) = A; \quad M \cap (-M) = \mathfrak{p} \text{ is a prime ideal in } A.$$

Proposition 3.4.
(a) Suppose T is a generating preprime and M is a maximal proper T-module, then $M \cup (-M) = A$.

(b) Suppose T is a preordering and M a maximal proper T-module then $\mathfrak{p} = M \cap (-M)$ is a prime ideal.

(c) Therefore: if T is a preordering and M is a maximal proper T-module then M is a semi-ordering.

Proof. Similar to proof in the preordering case
(a) Let $a \in A$, $a \notin M \cup (-M)$.
By maximality of M, we have:

$-1 \in (M + aT)$ and $-1 \in (M - aT)$.

Therefore, $-1 = s_1 + at_1$ and $-1 = s_2 - at_2$; for some $s_1, s_2 \in M$ and $t_1, t_2 \in T$.

This implies $-at_1 = 1 + s_1$ and $at_2 = 1 + s_2$.

So $-at_1t_2 = t_2 + s_1t_2$ and $at_2t_1 = t_1 + s_2t_1$.

So $0 = t_2 + t_1 + s_1t_2 + t_1s_2$.

So $-t_1 = t_2 + s_1t_2 + t_1s_2 \in M$.

Now since T is generating, pick $t_3, t_4 \in T$ such that $a = t_3 - t_4$, then

$-1 = s_1 + at_1 = s_1 + (t_3 - t_4)t_1 = s_1 + t_1t_3 + t_4(-t_1) \in M$. This is a contradiction.

(b) $\mathfrak{p} = M \cap -M$.

Clearly $\mathfrak{p} + \mathfrak{p} \subseteq \mathfrak{p}, -\mathfrak{p} = \mathfrak{p}, 0 \in \mathfrak{p}, T\mathfrak{p} \subseteq \mathfrak{p}$.

Since $A = T - T \Rightarrow A\mathfrak{p} \subseteq \mathfrak{p}$. Thus $\mathfrak{p}$ is an ideal.

So far we have only used that T is a generating preprime, to show that $\mathfrak{p}$ is a prime ideal we need that T is preordering:

Suppose $ab \in \mathfrak{p}, a \notin \mathfrak{p}$. Without loss of generality assume $a \notin M$.

Now this implies: $-1 \in M + aT$, so $-1 = s + at$; $s \in M, t \in T$

$\Rightarrow -b^2 = sb^2 + ab^2t \in M + \mathfrak{p} \subseteq M$.

Now since $b^2 \in T \subseteq M$, this implies $b^2 \in M \cap -M = \mathfrak{p}$.

So we are reduced to showing: $b^2 \in \mathfrak{p} \Rightarrow b \in \mathfrak{p}$.

Suppose $b^2 \in \mathfrak{p}, b \notin \mathfrak{p}$. Without loss of generality $b \notin M$.

Thus $-1 = s + bt$, for some $s \in M$ and $t \in T$.

So $1 + 2s + s^2 = (1 + s)^2 = (-bt)^2 = b^2t^2 \in \mathfrak{p} = M \cap -M$.

Thus $-1 = 2s + s^2 + \underbrace{(-b^2t^2)}_{(\in M)} \in M$, a contradiction since $-1 \notin M$.

(c) Clear. □

Our next aim is to show that under the additional assumption: "M is archimedian", then a maximal proper module M over a preordering is an ordering not just a semi-ordering. This is crucial in proof of Kadison-Dubois.