REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. ALGEBRAIC INDEPENDENCE AND TRANSCENDENCE DEGREE

Definition 1.1. (Recall) Let E/F be a field extension:

(1) $A \subseteq E$ is called **algebraically independent** over F if $\forall\ a_1, \ldots, a_n \in A$ there exists no nonzero polynomial $f \in F[X_1, \ldots, X_n]$ s.t. $f(a_1, \ldots, a_n) = 0$.

(2) $A \subseteq E$ is called a **transcendence basis** of E/F if A is a maximal subset (w.r.t. inclusion) of E which is algebraically independent over F.

Lemma 1.2. Let E/F be a field extension.

(1) (Steinitz exchange) $S \subseteq E$ is algebraically independent over F iff $\forall\ s \in S :\ s$ is transcendental over $F(S - \{s\})$ $\big($the subfield of E generated by $S - \{s\}\big)$.

(2) $S \subseteq E$ is a transcendence base for E/F iff S is algebraically independent over F and E is algebraic over $F(S)$. □

Theorem 1.3. The extension E/F has a transcendence base and any two transcendence bases of E/F have the same cardinality.

Proof. The existence follows by Zorn's lemma and the second statement uses the Steinitz exchange lemma (above). □

Definition 1.4. The cardinality of a transcendence base of E/F is called the **transcendence degree** of E/F, denoted by $\mathrm{trdeg}_F(E)$.

2. KRULL DIMENSION OF A RING

Definition 2.1 Let A be a commutative ring with 1.

(1) A **chain** of prime ideals of A is of the form
$\{0\} \subseteq \wp_0 \subsetneq \wp_1 \subsetneq \ldots \subsetneq \wp_k \subsetneq \ldots \subsetneq A$, where $\wp_i$ are prime ideals of A.

(2) The **Krull dimension** of A, denoted by dim (A) is defined to be the maximum k such that there is a chain of prime ideals of length k in A, i.e. $\wp_0 \subsetneq \wp_1 \subsetneq \ldots \subsetneq \wp_k$ $\left[\dim(A) \text{ can be infinite if arbitrary long chains}\right]$.

Theorem 2.2. Let F be a field and I be any prime ideal in $F[\underline{X}]$. Then

$$\dim\left(\frac{F[\underline{X}]}{I}\right) = \mathrm{trdeg}_F\left(ff\left(\frac{F[\underline{X}]}{I}\right)\right).$$

□

Recall 2.3. For $S \subseteq F^n$
$$\mathcal{I}(S) = \{f \in F[\underline{X}] \mid f(\underline{x}) = 0, \forall\ \underline{x} \in S\}$$
is the ideal of polynomials vanishing on S.

Definition 2.4. Dimension of semi-algebraic sets $\subseteq \mathbb{R}^n$: Let $K \subseteq \mathbb{R}^n$ be a semi-algebraic set. Then
$$\dim\ (K) := \dim\left(\frac{\mathbb{R}[\underline{X}]}{\mathcal{I}(K)}\right).$$

In the last lecture, we proved the following proposition:

Proposition 2.5. Suppose $n \geq 3$. Let $S = \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{X}]$ such that $\mathrm{int}(K_S) \neq \emptyset$. Then there exists $f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.

This is just a special case of the following result due to Scheiderer:

Theorem 2.6. Let S be a finite subset of $\mathbb{R}[\underline{X}]$ and $K_S \subseteq \mathbb{R}^n$ s.t. $\dim K_S \geq 3$. Then there exists $f \in \mathbb{R}[\underline{X}]$; $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.

To deduce Proposition 2.5 using Theorem 2.6 it suffices to prove the following lemma:

Lemma 2.7. Let $K \subseteq \mathbb{R}^n$ be a semi algebraic subset. Then

$$\text{int}(K) \neq \phi \Rightarrow \dim\,(K) = n$$

Proof. We **claim** that $\mathcal{I}(K) = \{0\}$:
$f \in \mathcal{I}(K) \Rightarrow f = 0$ on $K \Rightarrow f = 0$ on $\underbrace{\text{int}\,(K)}_{(\neq\phi)} \Rightarrow f$ vanishes on a nonempty open set $\Rightarrow f \equiv 0$ (by Remark 2.2 of lecture 2).
So, $\dim\,(K) = \dim\,(\mathbb{R}[\underline{X}]) = \text{trdeg}_F(\mathbb{R}(\underline{X}) = n$.

□

3. LOW DIMENSIONS

Proposition 3.1. Let $n = 2$, $K_S \subseteq \mathbb{R}^2$ and K_S contains a 2-dimensional cone. Then $\exists\ f \in \mathbb{R}[X, Y]$; $f \geq 0$ on $\mathbb{R}^2$; $f \notin T_S$.

Definition 3.2. (For $n = 1$) Let K be a basic closed semi algebraic subset of $\mathbb{R}$. Then K is a finite union of intervals.
The **natural description** S of K as a basic closed semi algebraic subset is defined as

1. if $a \in \mathbb{R}$ is the smallest element of K, then take $X - a \in S$
2. if $a \in \mathbb{R}$ is the greatest element of K, then take $a - X \in S$
3. if $a, b \in K$, $a < b$, $(a, b) \cap K = \phi$, then take $(X - a)(X - b) \in S$
4. no other polynomial should be in S.

Proposition 3.3. Let $K \subseteq \mathbb{R}$ be a non-empty basic closed semi algebraic subset and S is the natural description of K. Then $\forall\ f \in \mathbb{R}[X]$:

$$f \geq 0 \text{ on } K \Leftrightarrow f \in T_S,$$

i.e. for every basic semi algebraic subset K of $\mathbb{R}$, there exists a description S (namely the natural) so that T_S is saturated.

Proposition 3.4. Let $K \subseteq \mathbb{R}$ be a non-compact basic semi algebraic subset and S' be a description of K. Then

$$T_{S'} \text{ is saturated} \Leftrightarrow S' \supseteq S \text{ (up to a scalar multiple factor).}$$

Remark 3.5. Summarizing:

(1) $\dim(K_S) \geq 3 \Rightarrow T_S$ is not saturated.

(2) $\dim(K_S) = 2 \Rightarrow T_S$ can be or cannot be saturated (depending on the geometry of K and S).

(3) $\dim(K_S) = 1 \Rightarrow T_S$ can be or cannot be saturated [but depends on K and description S of K).

After all this discussion about positive polynomials, strictly positive polynomials, we now want to show **Schmüdgen's Positivstellensatz**:

Theorem 3.6. (Schmüdgen's Positivstellensatz) Let $S = \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[X_1, \ldots, X_n]$ and $K_S \subseteq \mathbb{R}^n$ be a compact non-empty basic closed semi algebraic set. And let $f \in \mathbb{R}[\underline{X}]$ s.t. $f > 0$ on K_S. Then $f \in T_S$.

Note that this holds for every finite description S of K.

To prove this we first need Representation Theorem (Stone-Krivine, Kadison-Dubois), which will be proved in the next lecture.