REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. RING OF FORMAL POWER SERIES

Definition 1.1. (Recall) Let $S = \{g_1, \dots, g_s\} \subseteq \mathbb{R}[X_1, \dots, X_n]$, then

$\mathbf{K_S} := \big\{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \ \forall \ i = 1, \dots, s\big\}$,

$\mathbf{T_S} := \Big\{ \sum_{e_1,\dots,e_s \in \{0,1\}} \sigma_e \ g_1^{e_1} \dots g_s^{e_s} \mid \sigma_e \in \Sigma\mathbb{R}[\underline{X}]^2, e = (e_1, \dots, e_s) \Big\}$ is the preordering generated by S.

Proposition 1.2. Let $n \geq 3$. Let S be a finite subset of $\mathbb{R}[\underline{X}]$ such that $K_S \subseteq \mathbb{R}^n$ has non empty interior. Then $\exists \ f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.

To prove proposition 1.2 we need to learn a few facts about formal power series rings:

Definition 1.3. $\mathbb{R}[[\underline{X}]] := \mathbb{R}[[X_1, \dots, X_n]]$ **ring of formal power series** in $\underline{X} = (X_1, \dots, X_n)$ with coefficients in $\mathbb{R}$, i.e. , $f \in \mathbb{R}[[\underline{X}]]$ is expressible uniquely in the form

$f = f_0 + f_1 + \dots,$

where f_i is a homogenous polynomial of degree i in the variables $X_1, \dots, X_n$.

Here:

- Addition is defined point wise, and
- multiplication is defined using distributive law:

$$\Big(\sum_{i=0}^{\infty} f_i\Big)\Big(\sum_{i=0}^{\infty} g_i\Big) = (f_0g_0)+(f_0g_1+f_1g_0)+(f_0g_2+f_1g_1+f_2g_0)+\ldots = \sum_{k=0}^{\infty}\Big(\sum_{i+j=k}(f_ig_j)\Big)$$

So, both addition and multiplication are well defined and $\mathbb{R}[[\underline{X}]]$ is an integral domain and $\mathbb{R}[\underline{X}] \subseteq \mathbb{R}[[\underline{X}]]$.

Notation 1.4. Fraction field of $\mathbb{R}[[\underline{X}]]$ is denoted by

$$ff(\mathbb{R}[[\underline{X}]]) := \mathbb{R}((\underline{X})).$$

The valuation $v : \mathbb{R}[[\underline{X}]] \to \mathbb{Z} \cup \{\infty\}$ defined by:

$$v(f) = \begin{cases} \text{least } i \text{ s.t. } f_i \neq 0 & \text{, if } f \neq 0 \\ \infty & \text{, if } f = 0 \end{cases}$$

extends to $\mathbb{R}((\underline{X}))$ via

$$v\left(\frac{f}{g}\right) := v(f) - v(g) \ .$$

Lemma 1.5. Let $f \in \mathbb{R}[[\underline{X}]]$; $f = f_k + f_{k+1} + \ldots$, where f_i homogeneous of degree i, $f_k \neq 0$. Assume that f is a sos in $\mathbb{R}[[\underline{X}]]$.
Then k is even and f_k is a sum of squares of forms of degree $\frac{k}{2}$.

Proof. $f = g_1^2 + \ldots + g_l^2$, and

$$g_i = g_{ij} + g_{i(j+1)} + \ldots \text{ , with } j = \min\{v(g_i)\ ;\ i = 1, \ldots, l\}$$

Then $f_0 = \ldots = f_{2j-1} = 0$ and $f_{2j} = \sum_{i=1}^{k} g_{ij}^2 \neq 0$

So, $k = 2j$. □

1.6. Units in $\mathbb{R}[[\underline{X}]]$: Let $f = f_0 + f_1 + \ldots$, with $v(f) = 0$ *i.e.* $f_0 \neq 0$. Then f factors as

$$f = a(1 + t);\ \text{ where } a \in \mathbb{R}^{\times},$$

$t \in \mathbb{R}[[\underline{X}]]$ and $v(t) \geq 1$. Indeed, set $a := f_0 \in \mathbb{R} \setminus \{0\}$; $t := \frac{1}{f_0}(f_1 + f_2 + \ldots)$
.

Lemma 1.7. $f \in \mathbb{R}[[\underline{X}]]$ is a unit of $\mathbb{R}[[\underline{X}]]$ if and only if $f_0 \neq 0$ (i.e. $v(f) = 0$).

Proof: $\dfrac{1}{1+t} = 1 - t + t^2 - \dots,$ for $t \in \mathbb{R}[[\underline{X}]]$; $v(t) \geq 1$
is a well defined element of $\mathbb{R}[[\underline{X}]]$.
So, if $v(f) = 0$, then $f = a(1+t)$ with $a \in \mathbb{R}^\times$ gives

$$f^{-1} = \frac{1}{a}\frac{1}{(1+t)} \in \mathbb{R}[[\underline{X}]].$$

□

Corollary 1.8. It follows that $\mathbb{R}[[\underline{X}]]$ is a local ring, with $I = \{f \mid v(f) \geq 1\}$ as its unique maximal ideal (the quotient is a field $\mathbb{R}$).

Lemma 1.9. Let $f \in \mathbb{R}[[\underline{X}]]$ a positive unit, i.e. $f_0 > 0$. Then f is a square in $\mathbb{R}[[\underline{X}]]$.

Proof. $f = a(1+t);\ a \in \mathbb{R}, a > 0, v(t) \geq 1$

$$\sqrt{f} = \sqrt{a}\sqrt{1+t}\ ,$$

where $\sqrt{1+t} := (1+t)^{1/2} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \dots$ is a well defined element of $\mathbb{R}[[\underline{X}]]$

□

Remark: For $u \in \mathbb{R}[[\underline{X}]]$ with $v(u) > 0$ (i.e. $u(\underline{0}) = 0$) and $\alpha \in \mathbb{R}$, one can define $(1+u)^\alpha := \sum_{n=0}^{+\infty} \frac{\alpha_n}{n!} u^n \in \mathbb{R}[[\underline{X}]]$ where $\alpha_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$. Then $p_u : \alpha \to (1+u)^\alpha$ is a group morphism $(\mathbb{R}, +) \to (\mathbb{R}[[\underline{X}]], \times)$ with $p_u(1) = 1 + u$.

Lemma 1.10. Suppose $n \geq 3$. Then $\exists\ f \in \mathbb{R}[\underline{X}]$ such that $f \geq 0$ on $\mathbb{R}^n$ and f is not a sum of squares in $\mathbb{R}[[\underline{X}]]$.

Proof. Let $f \in \mathbb{R}[\underline{X}]$ be any homogeneous polynomial which is ≥ 0 on $\mathbb{R}^n$ but is not a sum of squares in $\mathbb{R}[\underline{X}]$ (by Hilbert's Theorem such a polynomial exists). Now by lemma 1.5 it follows that f is not sos in $\mathbb{R}[[\underline{X}]]$. □

Now we prove Proposition 1.2:

Proof of Proposition 1.2. Let $S = \{g_1, \dots, g_s\}$

• W.l.o.g. assume $g_i \not\equiv 0$, for each $i = 1, \dots, s$. So $g := \prod_{i=1}^{s} g_i \not\equiv 0$

$\mathrm{int}(K_S) \neq \emptyset \Rightarrow \exists\ \underline{p} := (p_1, \dots, p_n) \in \mathrm{int}(K_S)$ with $g(\underline{p}) \neq 0$.
Thus $g_i(\underline{p}) > 0\ \forall\ i = i, \dots, s$.

• W.l.o.g. assume $\underline{p} = \underline{0}$ the origin

(by making a variable change $Y_i := X_i - p_i$, and noting that
$\mathbb{R}[Y_1, \dots, Y_n] = \mathbb{R}[X_1, \dots, X_n]$)

So $g_i(0, \dots, 0) > 0$ for each $i = i, \dots, s$ (i.e. has positive constant term),
that means $g_i \in \mathbb{R}[[\underline{X}]]$ is a positive unit in $\mathbb{R}[[\underline{X}]]\ \forall\ i = 1, \dots, s$.

By Lemma 1.9 (on positive units in power series): $g_i \in \mathbb{R}[[\underline{X}]]^2\ \forall\ i = i, \dots, s$.

So the preordering ${T_S}^A$ generated by $S = \{g_1, \dots, g_s\}$ in the ring $A := \mathbb{R}[[\underline{X}]]$ is just $\Sigma\mathbb{R}[[\underline{X}]]^2$.

Now using Lemma 1.10 : $\exists\ f \in \mathbb{R}[\underline{X}]$, $f \geq 0$ on $\mathbb{R}^n$ but f is not a sum of squares in $\mathbb{R}[[\underline{X}]]$ (i.e. $f \notin \Sigma\mathbb{R}[[\underline{X}]]^2 = T_S^A$) .

So $f \notin T_S = T_S^A \cap \mathbb{R}[[\underline{X}]]$. □(Proposition 1.2)

Proposition 1.2 that we just proved is just a special case of the following result due to Scheiderer:

Theorem 1.11. Let S be a finite subset of $\mathbb{R}[\underline{X}]$ such that K_S has dimension ≥ 3. Then $\exists\ f \in \mathbb{R}[\underline{X}]; f \geq 0$ on $\mathbb{R}^n$ and $f \notin T_S$.

To understand this result we need a reminder about dimension of semi algebraic sets from B5.

2. ALGEBRAIC INDEPENDENCE

Let E/F be a field extension:

Definition 2.1. (1) $a \in E$ is **algebraic** over F if it is a root of some non zero polynomial $f(X) \in F[X]$, otherwise a is a **transcendental** over F.

(2) $\{a_1, \dots, a_n\} \subseteq E$ is called **algebraically independent** over F if there is no nonzero polynomial $f(x_1, \dots, x_n) \in F[X_1, \dots, X_n]$ s.t. $f(a_1, \dots, a_n) = 0$.
In general $A \subseteq E$ is algebraically independent over F if every finite subset of A is algebraic independent over F.

(3) A **transcendence base** of E/F is a maximal subset (w.r.t. inclusion) of E which is algebraically independent over F.