REAL ALGEBRAIC GEOMETRY LECTURE NOTES
PART II: POSITIVE POLYNOMIALS
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Contents

1. PROOF OF HILBERT'S THEOREM (Continued)

Theorem 1.1. (Recall) (Hilbert) $\sum_{n,m} = \mathcal{P}_{n,m}$ iff

(i) $n = 2$ or

(ii) $m = 2$ or

(iii) $(n, m) = (3, 4)$.

And in all other cases $\sum_{n,m} \subsetneq \mathcal{P}_{n,m}$.

We have shown one direction ($\Leftarrow$) of Hilbert's Theorem (1.1 above), i.e. if $n = 2$ or $m = 2$ or $(n, m) = (3, 4)$, then $\sum_{n,m} = \mathcal{P}_{n,m}$. To prove the other direction we have to show that:

$$\sum_{n,m} \subsetneq \mathcal{P}_{n,m} \quad \forall (n, m) \text{ s.t. } n \geq 3, m \geq 4 \ (m \text{ even}) \text{ with } (n, m) \neq (3, 4). \tag{1}$$

Hilbert showed (using algebraic geometry) that $\sum_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\sum_{4,4} \subsetneq \mathcal{P}_{4,4}$. This is a reduction of the general problem (1), indeed we have:

Lemma 1.2. If $\sum_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\sum_{4,4} \subsetneq \mathcal{P}_{4,4}$, then

$$\sum_{n,m} \subsetneq \mathcal{P}_{n,m} \text{ for all } n \geq 3, m \geq 4 \text{ and } (n,m) \neq (3,4), \ (m \text{ even}).$$

Proof. Clearly, given $F \in \mathcal{P}_{n,m} \setminus \sum_{n,m}$, then $F \in \mathcal{P}_{n+j,\ m} \setminus \sum_{n+j,\ m}$ for all $j \geq 0$.

Moreover, we **claim:** $F \in \mathcal{P}_{n,m} \setminus \sum_{n,m} \Rightarrow x_1^{2i} F \in \mathcal{P}_{n,\ m+2i} \setminus \sum_{n,\ m+2i}$ $\forall\, i \geq 0$

Proof of claim: Assume for a contradiction that

for $i = 1$
$$x_1^2 F(x_1, \ldots, x_n) = \sum_{j=1}^{k} f_j^2(x_1, \ldots, x_n),$$

then L.H.S vanishes at $x_1 = 0$, so R.H.S also vanishes at $x_1 = 0$.
So $x_1 | f_j$ $\forall\, j$, so $x_1^2 | f_j^2$ $\forall\, i$. So, R.H.S is divisible by x_1^2. Dividing both sides by x_1^2 we get a sos representation of F, a contradiction since $F \notin \sum_{n,m}$. □

So we just need to show that: $\sum_{3,6} \subsetneq \mathcal{P}_{3,6}$, and $\sum_{4,4} \subsetneq \mathcal{P}_{4,4}$.

Hilbert described a method (non constructive) to produce counter examples in the 2 crucial cases, but no explicit examples appeared in literature for next 80 years.
In 1967 Motzkin presented a specific example of a ternary sextic form that is positive semidefinite but not a sum of squares.

2. THE MOTZKIN FORM

Proposition 2.1. The Motzkin form

$$M(x,y,z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \sum_{3,6}.$$

Proof. Using the arithmetic geometric inequality (Lemma 2.2 below) with $a_1 = z^6, a_2 = x^4y^2, a_3 = x^2y^4$ and $n = 3$, clearly gives $M \geq 0$.
Degree arguments give M is not a sum of squares □

Lemma 2.2. (Arithmetic-geometric inequality I) Let $a_1, a_2, \ldots, a_n \geq 0$; $n \geq 1$. Then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}.$$

Lemma 2.3. (Arithmetic-geometric inequality II) Let $\alpha_i \geq 0$, $a_i \geq 0$; $i = 1, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$.Then

$$\alpha_1 a_1 + \ldots + \alpha_n a_n - a_1^{\alpha_1} \ldots a_n^{\alpha_n} \geq 0$$

3. ROBINSON'S METHOD (1970)

In 1970's R. M. Robinson gave a ternary sextic based on the method described by Hilbert, but after drastically simplifying Hilbert's original ideas. He used it to construct examples of forms in $\mathcal{P}_{4,4} \setminus \sum_{4,4}$ as well as forms in $\mathcal{P}_{3,6} \setminus \sum_{3,6}$.

This method is based on the following lemma:

Lemma 3.1. A polynomial $P(x,y)$ of degree at most 3 which vanishes at eight of the nine points $(x,y) \in \{-1,0,1\} \times \{-1,0,1\}$ must also vanish at the ninth point.

Proof. Assign weights to the following nine points:

$$w(x,y) = \begin{cases} 1 & , \text{ if } x,y = \pm 1 \\ -2 & , \text{ if } (x = \pm 1, y = 0) \text{ or } (x = 0, y = \pm 1) \\ 4 & , \text{ if } x,y = 0 \end{cases}$$

Define the weight of a monomial as:

$$w(x^k y^l) := \sum_{i=1}^{9} w(q_i) x^k y^l (q_i) \ , \quad \text{for} \quad q_i \in \{-1,0,1\} \times \{-1,0,1\}$$

Define the weight of a polynomial $P(x,y) = \sum_{k,l} c_{k,l} \ x^k y^l$ as:

$$w(P) := \sum_{k,l} c_{k,l} \ w(x^k y^l) \qquad \text{for } c_{k,l} \in \mathbb{R}.$$

Claim 1: $w(x^k y^l) = 0$ unless k and l are both strictly positive and even.

<u>Proof of claim 1:</u> Let us compute the monomial weights

- if $k = 0, l \geq 0$: then we have
$$w(x^k y^l) = 1 + (-1)^l + 1 + (-1)^l + (-2) + (-2)(-1)^l = 0$$
- if $l = 0, k \geq 0$: then similarly we have $w(x^k y^l) = 0$, and
- if $k, l > 0$: then we have
$$w(x^k y^l) = 1 + (-1)^l + (-1)^k + (-1)^{k+l} = \begin{cases} 0 \ , & \text{if either } k \text{ or } l \text{ is odd} \\ 4 \ , & \text{otherwise} \end{cases}$$

$\square$ (claim 1)

Claim 2: $w(P) = \sum_{i=1}^{9} w(q_i)P(q_i)$

<u>Proof of claim 2:</u>
$$w(P) := \sum_{k,l} c_{k,l}\ w(x^k y^l) = \sum_{k,l} c_{k,l}\ \sum_{i=1}^{9} w(q_i)x^k y^l(q_i)$$
$$= \sum_{i=1}^{9} w(q_i) \sum_{k,l} c_{k,l} x^k y^l(q_i) = \sum_{i=1}^{9} w(q_i)P(q_i)$$

$\square$ (claim 2)

Now, claim 1 and definition of $w(P)$ $\Rightarrow$ if $\deg(P(x,y)) \leq 3$ then $w(P) = 0$.

Also, from claim 2 we get:
$P(1,1)+P(1,-1)+P(-1,1)+P(-1,-1)+(-2)P(1,0)+(-2)P(-1,0)+(-2)P(0,1)+(-2)P(0,-1)+4P(0,0) = 0$

Now verify that if $P(x,y) = 0$ for any eight (of the nine) points, then we are left with $\alpha P(x,y) = 0$ (for some $\alpha \neq 0$) at the ninth point. $\square$

4. THE ROBINSON FORM

Theorem 4.1. Robinsons form
$R(x,y,z) = x^6+y^6+z^6-(x^4y^2+x^4z^2+y^4x^2+y^4z^2+z^4x^2+z^4y^2)+3x^2y^2z^2$ is psd but not a sos, i.e. $R \in \mathcal{P}_{3,6} \setminus \sum_{3,6}$.

Proof. Consider the polynomial

$$P(x,y) = (x^2+y^2-1)(x^2-y^2)^2 + (x^2-1)(y^2-1) \tag{2}$$

Note that $R(x,y,z) = P_h(x,y,z) = z^6 P(x/z, y/z)$.

By our observation: P_h is psd iff P psd; P_h is sos iff P is sos,

We shall <u>show that</u> $P(x,y)$ is psd but not sos.

Multiplying both sides of (2) by (x^2+y^2-1) and adding to (2) we get:

$$(x^2+y^2)P(x,y) = x^2(x^2-1)^2 + y^2(y^2-1)^2 + (x^2+y^2-1)^2(x^2-y^2)^2 \tag{3}$$

From (3) we see that $P(x,y) \geq 0$, i.e. $P(x,y)$ is psd.

Assume $P(x,y) = \sum_j P_j(x,y)^2$ is sos

$\deg P(x,y) = 6$, so $\deg P_j \leq 3\ \forall\ j$.

By (2) it is easy to see that $P(0,0) = 1$ and $P(x,y) = 0$ for all other eight points $(x,y) \in \{-1,0,1\}^2 \setminus \{(0,0)\}$, therefore every $P_j(x,y)$ must also vanish at these eight points.

Hence by Lemma 3.1 (above) it follows that: $P_j(0,0) = 0 \;\forall\; j$.

So $P(0,0) = 0$, which is a contradiction. □

Proposition 4.2. The quarternary quartic $Q(x,y,z,w) = w^4+x^2y^2+y^2z^2+x^2z^2 - 4xyzw$ is psd, but not sos, i.e., $Q \in \mathcal{P}_{4,4} \setminus \sum_{4,4}$.

Proof. The arithmetic-geometric inequality clearly implies $Q \geq 0$.

Assume now that $Q = \sum_j q_j^2$, $q_j \in \mathcal{F}_{4,2}$.

Forms in $\mathcal{F}_{4,2}$ can only have the following monomials:

$x^2, y^2, z^2, w^2, xy, xz, xw, yz, yw, zw$

If x^2 occurs in some of the q_j, then x^4 occurs in q_j^2 with positive coefficient and hence in $\sum q_j^2$ with positive coefficient too, but this is not the case.

Similarly q_j does not contain y^2 and z^2.

The only way to write x^2w^2 as a product of allowed monomials is $x^2w^2 = (xw)^2$.

Similarly for y^2w^2 and z^2w^2.

Thus each q_j involves only the monomials xy, xz, yz and w^2.

But now there is no way to get the monomial $xyzw$ from $\sum_j q_j^2$, hence a contradiction.

□

Proposition 4.3. The ternary sextic $S(x,y,z) = x^4y^2+y^4z^2+z^4x^2-3x^2y^2z^2$ is psd, but not a sos, i.e., $S \in \mathcal{P}_{3,6} \setminus \sum_{3,6}$.

□